2011 LSMSA Math Competition
Advanced Math Individual

1. How many positive integers less than 100 contain a 7 as a digit?

   Solution: Integers less than 100 that contain 7 as a digit:

   $$7, 17, 27, 37, 47, 57, 67, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 87, 97.$$

   Counting, we find there are 19 such numbers.

2. Evaluate $\cos(\sin(\tan(0)))$.

   Solution: $\cos(\sin(\tan(0))) = \cos(\sin(0)) = \cos(0) = 1$.

3. Find the height of an equilateral triangle of side length 1.

   Solution: The height separates the triangle into 2 equal right triangles. Using the Pythagorean Theorem: $h^2 + \left(\frac{1}{2}\right)^2 = 1^2$. Solving for h: $h = \frac{\sqrt{3}}{2}$.

4. Evaluate $\sum_{i=-2010}^{2011} i$.

   Solution: $\sum_{i=-2010}^{2011} i = \sum_{i=-2010}^{2010} i + 2011 = 2011$.

5. Find the sum of all factors of 72.

   Solution: The factors of 72 are: 72, 36, 24, 18, 12, 9, 8, 6, 4, 3, 2, 1. Adding them results in 195.

6. Find the loci of points equidistant from (1,0) and (0,1).

   Solution: The loci of points equidistant from two points is a line perpendicular to a line connecting the two points and intersecting at the two points' midpoint. For (1,0) and (0,1), the line is $y = x$.

7. Assume the radius of the Earth is R. The Flash races Superman around the world. The Flash is arrogant and runs around the equator, while Superman runs around the 30° latitude line. The Flash runs at speed F, while Superman runs at $\frac{2}{3}F$. Who wins and what is the ratio of the loser's time to the winner's?

   Solution: In order to determine the time it took for each superhero to finish the race. The distance Flash runs is $2\pi R$. The radius of the circle Superman runs is $\frac{R\sqrt{3}}{2}$ so the distance Superman runs is $\pi R\sqrt{3}$. The time it took Flash to finish the race is $t_F = \frac{2\pi R}{F}$. The time it took Superman to finish the race is $t_S = \frac{3\pi R\sqrt{3}}{2F}$. Since $t_F > t_S$ so Superman wins the race. $\frac{t_F}{t_S} = \frac{2\pi R}{F}\frac{2F}{3\pi R\sqrt{3}} = \frac{4}{3\sqrt{3}}$.

8. What is the probability of flipping a normal coin 5 times and getting 2 heads and 3 tails in any order?

   Solution: First we find the probability of flipping the coins in a specific order, for example 2 heads and 3 tails. That probability is $\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2} = \frac{1}{2^5}$. Next, multiply this by the number of possible ways to order the 2 heads and 3 tails. This is $\begin{pmatrix} 5 \\ 2 \end{pmatrix} = 10$. Therefore the probability of flipping the coins in any order is $\frac{10}{2^5}$ or $\frac{5}{16}$.

9. A parabolic bridge spans 6 feet with a maximum height of 18 feet. Suddenly a flash flood causes the water to rise 10 feet above ground level, so the top of the bridge is only 8 feet above the water. What is the distance spanned by the bridge that is above water?

   Solution: The equation for the bridge is $y = -2(x-3)(x+3)$. Simplifying: $y = -2x^2 + 18$. The equation for the bridge after the water rises is $y = -2x^2 + 8$ or $y = -2(x-2)(x+2)$. Therefore the distance spanned by the bridge is 4 feet.

10. Evaluate $(1-i)^{24}$.

    Solution: $(1-i)^{24} = \left((1-i)^2\right)^{12} = (1-2i-1)^{12} = (-2i)^{12} = (-2)^{12}i^{12} = 2^{12} = 4096$.

11. Mario is on a 1×1 cubic planet and wants to reach the princess on the opposite corner. How far must he move? Note that he can't go through the planet – he has to always be touching one of the cube's faces.

    Solution: Since Mario can only go along the planet's surface, it is useful to represent the planet as 6 squares sewn together, as though you had unfolded the cube. From here, it is plain to see that the corner which Mario must reach is 2 units away horizontally and 1 unit away vertically. Applying the Pythagorean Theorem results in the distance being $\sqrt{5}$.

12. Find the next 5 in the sequence

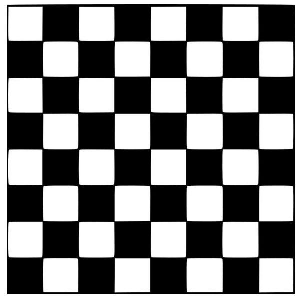
    $$4, 6, 7, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19, \ldots$$

    Solution: These are the numbers that are "missing:" 1, 2, 3, 5, 8, 13. This is the Fibonacci sequence. This means the next 5 numbers in the sequence are 20, 22, 23, 24, 25, since 21 is the next number in the Fibonacci sequence.

13. If $A + B = 7$ and $AB = 3$, find $A^2 + B^2$.

    Solution: First square both sides of $A + B = 7$ to obtain $A^2 + AB + B^2 = 49$. Next subtract 2 times the second equation from both sides: $A^2 + AB + B^2 - 2AB = 49 - 2 \cdot 3$. Simplifying: $A^2 + B^2 = 43$.

14. How many squares are on a chess board? (Remember to count the 2×2, 3×3, etc.)



Solution: Start counting and you will find that there are 8^2 small squares. The 2×2 squares are one unit wider and one unit taller than the 1×1's, so you can fit only $8 - 1 = 7$ across and $8 - 1 = 7$ vertically. Therefore, there are 7^2 squares of size 2×2. Similarly, there will be 6^2 squares of size 3×3 and so on until you find that there is only 1 square of size 8×8. The problem then becomes adding all of theses squares: $8^2 + 7^2 + 6^2 + 5^2 + 4^2 + 3^2 + 2^2 + 1^2$. This can be written as: $\sum_{i=1}^{8} i^2$. This formula for finding the sum of the first n squares is $\frac{n(n+1)(2n+1)}{6}$. Using this the answer is obtained: $\frac{8(8+1)(2\cdot 8+1)}{6} = 204$.

15. Perform the sum $\sum\limits_{n=1}^{\infty} \frac{n}{2^n}$.

Solution: $\sum\limits_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots = \left(\frac{1}{2}\right) + \left(\frac{1}{2}\frac{1}{2}\right) + \left(\frac{1}{2}\frac{1}{2}\frac{1}{2}\right) + \cdots = \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots\right) + \left(\frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots\right) + \left(\frac{1}{2^3} + \frac{1}{2^4} + \cdots\right) + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n} + \sum_{n=2}^{\infty} \frac{1}{2^n} + \sum_{n=3}^{\infty} \frac{1}{2^n} + \cdots = \frac{1/2}{1-1/2} + \frac{1/4}{1-1/2} + \frac{1/8}{1-1/2} + \cdots = 2 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 2 \cdot \frac{1}{8} + \cdots = 2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right) = 2\left(\frac{1/2}{1-1/2}\right) = 2.$