2011 LSMSA Math Competition

Advanced Math Team

1. Find $\cos(3\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$.

   Solution: $\cos(3\theta) = \cos(\theta+2\theta) = \cos(\theta)\cos(2\theta)-\sin(\theta)\sin(2\theta) = \cos(\theta)[\cos^2(\theta)-\sin^2(\theta)]-\sin(\theta)[2\sin(\theta)\cos(\theta)] = \cos(\theta)[\cos^2(\theta) - 3\sin^2(\theta)]$.

2. If $A+B+C+D+E = 45$ and A, B, C, D, and E are in arithmetic progression ($B = r+A$, $C = r+B$, etc.), which of A, B, C, D, and E can you find and what is its value?

   Solution: B is r less than C and D is r more than C, so B and D have an average of C. That is $\frac{B+D}{2} = C$ or $B + D = 2C$. Similarly, since A is $2r$ less than C and E is $2r$ more than C, $A + E = 2C$. Now $A + B + C + D + E = 2C + 2C + C = 5C$. Then $5C = 45$ and $C = 9$.

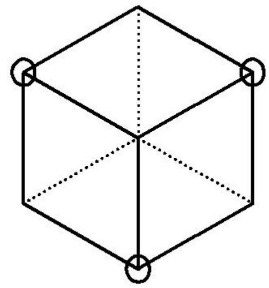
3. On 1B, there is a 0.4 probability that you'll hear a noise and, if there is a noise, there is a 0.9 probability that David Wolff is being reckless. If no noise is heard, there is a 0.4 probability that David is being reckless. If David's being reckless, what is the probability that noise is made?

   Solution: The probability of David being reckless and noise being made is 0.9(0.4) and the probability of David being reckless and noise not being made is 0.6(0.4). Therefore the probability that noise is being made given that David is being reckless is $\frac{0.4(0.6)}{0.9(0.4)+0.6(0.4)} = 0.6$.

4. Mario has legs of length 1 meter and takes one step per second, with each step creating a 30° angle between his legs. He eats a mushroom, changes color, and grows to 5x his original height (meaning his legs are now 5 times longer). Also, he takes longer steps so at the end of each step, there is a 60° angle between his legs. Still, he takes one step per second. What is the ratio of his new speed to his old speed?

   Solution: The formula for speed is $\frac{d}{t} = r$, and since a step takes Mario the same amount of time whether he's big or small, the ratio between his big speed and his small speed is the same as the ratio between the size of one of his big steps and small steps. Using the las of cosines, the length of Mario's small step is $C^2 = A^2 + B^2 - 2AB\cos(c)$ where A and B are the lengths of hs legs, C is the length of his step and c is the 30° angle. $C^2 = 1 + 1 - 2\cos(30°) = 2 - \sqrt{3}$. So $C = \sqrt{2-\sqrt{3}}$ Mario's bigger step form an equilateral triangle, so we know that the step size is 5, the same as the length of his legs. So the fraction of Mario's small speed to his big speed is $\frac{\sqrt{2-\sqrt{3}}}{5}$.

5. A cube is going through a planar force-field. The three circled corners are in the plane of the force-field and the cross sectional area of the triangle found by these 3 points is $\frac{3\sqrt{3}}{4}$. Find the volume of the cube.



Solution: The formula for the area of an equilateral triangle is $\frac{s^2\sqrt{3}}{4}$, so each side of the triangle must be $\sqrt{3}$. Then, notice that these sides of the triangle are also the diagonals of one of the faces of the cube. You can use Pythagorean Theorem to discover that the side length of the cube is $\frac{\sqrt{3}}{\sqrt{2}}$, so the volume of the cube is $\left(\frac{\sqrt{3}}{\sqrt{2}}\right)^3 = \frac{3\sqrt{3}}{2\sqrt{2}} = \frac{3\sqrt{6}}{4}$.

6. Find the Greatest Common Factor of all numbers in the form $abab$ where a and b are digits. (For example 4343)

   Solution: Expand $abab$ to get $1000a + 100b + 10a + b$, which can be re-arranged to $1010a + 101b$, or $101 \cdot 10a + 101b$, which can be factored as $101 \cdot (10a + b)$. From this, it is clear that 101 is the Greatest Common Factor.

7. Evaluate $(\cos(15°) - \sin(15°))^{10}$.

   Solution: $(\cos(15°) - \sin(15°))^{10} = \left((\cos(15°) - \sin(15°))^2\right)^5 = (\cos^2(15°) - 2\cos(15°)\sin(15°) + \sin^2(15°))^5 = (1 - 2\sin(15°)\cos(15°))^5 = (1 - \sin(30°))^5 = (1 - 0.5)^5 = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$.

8. A path of length $95\frac{1}{3}$ and width $1\frac{1}{2}$ spirals around and completely covers a rectangular garden. What are the dimensions of the garden? (The dimensions are integers)

   Solution: The key is that the path completely covers the garden, so the path has the same area as the garden. The area of the path is $95\frac{1}{3} \cdot 1\frac{1}{2} = 143$. Factoring 143 leads to the dimensions of the garden: 11×13.

9. A rectangular prism has an edge length of k and a surface area of m. In terms of k and m , what is the distance from one corner of the prism to the opposite corner?

   Solution: Let the sides of the prism be a, b, c. The edge length is then $k = 4a + 4b + 4c$, and the surface area is $m = 2ab + 2ac + 2bc$. The distance from one corner to the opposite corner can be written as $x = \sqrt{a^2 + b^2 + c^2}$ or $x^2 = a^2 + b^2 + c^2$ where x is the distance from one corner to the opposite corner. First, we take the edge-length equation and divide by 4 to get $\frac{1}{4}k = a + b + c$. The next step is to square both sides to obtain $\frac{1}{16}k^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$. Finally, if we subtract the surface area equation from this we are left with $\frac{1}{16}k^2 - m = a^2 + b^2 + c^2$, which is equal to x^2, so $x = \sqrt{\frac{1}{16}k^2 - m}$.

10. Prove by induction: $\frac{n^5 - n}{10}$ is an integer.

    Solution: Show that it is true for $n = 1$: $\frac{1^5 - 1}{10} = 0$ and 0 is an integer. Next, assume that for some value k, the statement is true. Suppose that for some k, $\frac{k^5 - k}{10}$ is in integer. The final step is to show that the statement holds true for $k+1$: $\frac{(k+1)^5-(k+1)}{10}$ is an integer. Using the binomial theorem, this is $\frac{k^5+5k^4+10k^3+10k^2+5k+1-k-1}{10}$. This is $\frac{10k^3+10k^2}{10}+\frac{k^5-5}{10}+\frac{5k^4+5k}{10}$. Simplifying: $k^3+k^2+\frac{k^5-k}{10}+\frac{5(k^4+k)}{10}$. We need to prove that $\frac{5(k^4+k)}{10}$ is an integer. This will be true if and only if k^4+k is even. If k is even then k^4 is also even so the sum $k+k^4$ is even. If k is odd, then k^4 is also odd and the sum of two odd numbers is even.

11. Write a formula that relates the prime factorization of any number to the sum of that number's factors. For example the sum of factors of 72 in terms of its prime factors: 2^3 and 3^2.

    Solution: The sum equals $\prod_{i=1}^{n}\sum_{j=1}^{1+m_i} i^j$ where n is the number of unique prime factors the numbers has and m_i is the number of times the ith prime factor appears in the prime factorization. i goes through the number's unique prime factors.

12. "How many children have you, and how old are they?" asked the guest, a mathematics teacher.

    "I have three boys," said Mr. Smith. "The product of their ages is 72 and the sum of their ages is the street number."

    The guest went to look at the entrance, came back and said:

    "The problem is indeterminate."

    "Yes, that is so," said Mr. Smith, "but I still hope that the oldest boy will some day win the Stanford competition."

    Tell the ages of the boys, stating your reasons.

    Solution: Since the product of the ages is 72, the ages must be composed of the numbers 3, 3, 2, 2, 2, 1. We are told that, given the sum of the children's ages, you still can't tell how old they are. That means, we need to find a way to combine 3, 3, 2, 2, 2, 1 in two different ways so that the sum of the numbers is the same. A hit is that the fact that there is an oldest boy makes the problem possible. This means that if we find the sum such that there are two oldest children, it will be the same as the sum of the kid's actual ages. $3\cdot 2+3\cdot 2+2\cdot 1=14$. Then, we find a way to combine the numbers such that there is an oldest child and the sum of the ages is still 14: $2\times 2\times 2+3+3\times 1=14$, so their ages are 8, 3, and 3.

13. Prove the identity

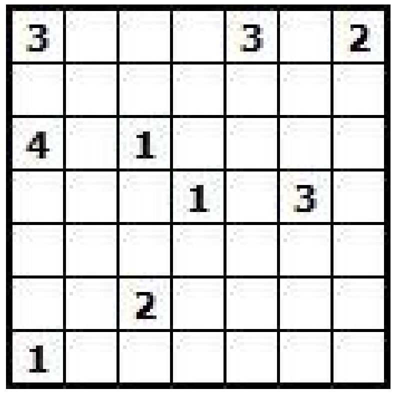
$$\cos\frac{\alpha}{2}\cos\frac{\alpha}{4}\cos\frac{\alpha}{8}=\frac{\sin\alpha}{8\sin\frac{\alpha}{8}}$$

    Solution: On the left hand side multiply by $\frac{8\sin\left(\frac{\alpha}{8}\right)}{8\sin\left(\frac{\alpha}{8}\right)}$ to get $\frac{\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{4}\right)\cos\left(\frac{\alpha}{8}\right)8\sin\left(\frac{\alpha}{8}\right)}{8\sin\left(\frac{\alpha}{8}\right)}$. We can rearrange terms to get $\frac{\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{4}\right)4\left(2\sin\left(\frac{\alpha}{8}\right)\cos\left(\frac{\alpha}{8}\right)\right)}{8\sin\left(\frac{\alpha}{8}\right)}$. Next, we apply the double angle identity to obtain $\frac{\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{4}\right)4\sin\left(\frac{\alpha}{4}\right)}{8\sin\left(\frac{\alpha}{8}\right)}$. We rearrange again: $\frac{\cos\left(\frac{\alpha}{2}\right)2\left(2\sin\left(\frac{\alpha}{4}\right)\cos\left(\frac{\alpha}{4}\right)\right)}{8\sin\left(\frac{\alpha}{8}\right)}$, and use the double angle identity again: $\frac{\cos\left(\frac{\alpha}{2}\right)2\sin\left(\frac{\alpha}{2}\right)}{8\sin\left(\frac{\alpha}{8}\right)}$. Again, we rearrange: $\frac{2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)}{8\sin\left(\frac{\alpha}{8}\right)}$ and use the double angle identity one more time to get: $\frac{\sin\alpha}{8\sin\frac{\alpha}{8}}$.

14. Ten people are sitting around a round table. The sum of ten dollars is to be distributed among them according to the rule that each person receives one half of the sum that his two neighbors receive jointly. Is there just one way to distribute the money? Explain.

Solution: If each person has the average of his neighbors, then the people's money will increase in an arithmetic sequence. Person one will get k dollars, person two, $k+r$ dollars, person three, $k+2r$, and so on until person ten gets $k+9r$ dollars. From this, it is clear that person one will not have the average of person ten and person two unless $r=0$ because $\frac{(k+9r)+(k+r)}{2}=k \Rightarrow r=0$, so everyone must have the same amount of money: \$1.

15. Solve the following Nurikabe Puzzle:



Solution: