Differential Equations Project

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Problem: (p 104, #8)
Show that the half-life of a radioactive substance is, in general,

$$t = \frac{(t_2 - t_1)\ln(2)}{\ln(A_1/A_2)},$$

where $A_1 = A(t_1)$ and $A_2 = A(t_2)$, $t_1 < t_2$.

Solution:
A radioactive substance's rate of decay is proportional to the amount of substance present at any time. From this we get a generic form of the equation:

$$\frac{dA}{dt} = kA \tag{1}$$

We are given that $A_1 = A(t_1)$ and $A_2 = A(t_2)$, $t_1 < t_2$. It is also useful to define $A(0)$ as A_0.
Rearranging and then integrating equation (1) yields the following:

$$\frac{dA}{A} = kdt$$

$$\int \frac{dA}{A} = \int k\,dt$$

$$\ln(A) = kt + C_1$$

$$e^{\ln(A)} = e^{kt+C_1}$$

$$A = e^{C_1}e^{kt}$$

$$A = Ce^{kt} \tag{2}$$

Assuming that $A(0) = A_0$,

$$A_0 = Ce^{k(0)}$$

$$C = A_0$$

$$A = A_0e^{kt} \tag{3}$$

Plugging in $A(t_1) = A_1$ and $A(t_2) = A_2$ yields:

$$A_1 = A_0e^{kt_1} \text{ and } A_2 = A_0e^{kt_2}$$

$$A_0 = \frac{A_1}{e^{kt_1}} \text{ and } A_0 = \frac{A_2}{e^{kt_2}}$$

Combining yields:

$$\frac{A_1}{e^{kt_1}} = \frac{A_2}{e^{kt_2}}$$

Solving for k yields:

$$\ln\left(\frac{A_1}{e^{kt_1}}\right) = \ln\left(\frac{A_2}{e^{kt_2}}\right)$$

$$\ln(A_1) - \ln(e^{kt_1}) = \ln(A_2) - \ln(e^{kt_2})$$

$$-kt_1 + kt_2 = \ln(A_2) - \ln(A_1)$$

$$k(t_2 - t_1) = \ln\left(\frac{A_2}{A_1}\right)$$

$$k = \frac{\ln\left(\frac{A_2}{A_1}\right)}{(t_2 - t_1)}$$

$$A = A_0 e^{\frac{\ln\left(\frac{A_2}{A_1}\right)}{(t_2 - t_1)} t} \tag{4}$$

The half-life of a substance is the time it takes for half of the substance to disappear $A(t_{1/2}) = \frac{A_0}{2}$. Plugging this into equation (4) and solving for $t_{1/2}$ yields:

$$\frac{A_0}{2} = A_0 e^{\frac{\ln\left(\frac{A_2}{A_1}\right)}{(t_2 - t_1)} t_{1/2}}$$

$$\frac{1}{2} = e^{\frac{\ln\left(\frac{A_2}{A_1}\right)}{(t_2 - t_1)} t_{1/2}}$$

$$\ln\left(\frac{1}{2}\right) = \frac{\ln\left(\frac{A_2}{A_1}\right)}{(t_2 - t_1)} t_{1/2}$$

$$-\ln(2) = \frac{\ln\left(\frac{A_2}{A_1}\right)}{(t_2 - t_1)} t_{1/2}$$

$$\frac{\ln(2)(t_2 - t_1)}{(-1)\ln\left(\frac{A_2}{A_1}\right)} = t_{1/2}$$

$$\frac{\ln(2)(t_2 - t_1)}{\ln\left(\left(\frac{A_2}{A_1}\right)^{-1}\right)} = t_{1/2}$$

$$\frac{\ln(2)(t_2 - t_1)}{\ln\left(\frac{A_1}{A_2}\right)} = t_{1/2}$$

or

$$t = \frac{(t_2 - t_1)\ln(2)}{\ln(A_1/A_2)} \tag{5}$$

Problem: (p 104, #12)

A thermometer is taken from an inside room to the outside where the air temperature is 5°F. After 1 minute the thermometer reads 55°F, and after 5 minutes the reading is 30°F. What is the initial temperature of the room.

Solution:

$$\frac{dT}{dt} = k(T - T_m) \tag{6}$$

We know the following:

$$T_m = 5°\text{F},\ T(1) = 55°\text{F},\ T(5) = 30°\text{F}.$$

It is also useful to define $T(0) = T_0$. From Equation (6) we get:

$$\frac{dT}{dt} = k(T - 5)$$
$$\frac{dT}{T-5} = kdt$$
$$\int \frac{dT}{T-5} = \int k\,dt$$
$$ln(T-5) = kt + C_1$$
$$T - 5 = e^{kt+C_1}$$
$$T = Ce^{kt} + 5 \tag{7}$$

Plugging in $T(0) = T_0$ yields the following:

$$T_0 = Ce^{k(0)} + 5$$
$$C = T_0 - 5$$
$$T = (T_0 - 5)e^{kt} + 5 \tag{8}$$

Plugging in $T(1) = 55$ and $T(5) = 30$ yields the following:

$$55 = (T_0 - 5)e^k + 5 \text{ and } 30 = (T_0 - 5)e^{k(5)} + 5$$
$$50 = (T_0 - 5)e^k \text{ and } 25 = (T_0 - 5)e^{5k}$$
$$\frac{50}{T_0 - 5} = e^k \text{ and } \frac{25}{T_0 - 5} = e^{5k}$$
$$\ln\left(\frac{50}{T_0 - 5}\right) = k \text{ and } \ln\left(\frac{25}{T_0 - 5}\right) = 5k$$

Subtracting the two equations yields:

$$\ln\left(\frac{25}{T_0 - 5}\right) - \ln\left(\frac{50}{T_0 - 5}\right) = 5k - k$$
$$\ln\left(\frac{25}{T_0 - 5} \times \frac{T_0 - 5}{50}\right) = 4k$$
$$\ln(0.5) = 4k$$
$$k = \frac{\ln(0.5)}{4}$$

Plugging this k into $\ln(50/(T_0 - 5)) = k$ yields:

$$\ln\left(\frac{50}{T_0 - 5}\right) = \frac{\ln(2)}{4}$$
$$\frac{50}{T_0 - 5} = e^{\ln(0.5)/4}$$
$$T_0 = \frac{50}{e^{\ln(0.5)/4}} + 5 \approx 64.46035575°\text{F}$$

Problem: (p 106, #26)

Beer containing 6% alcohol per gallon is pumped into a vat that initially contains 400 gallons of beer at 3% alcohol. The rate at which the beer is pumped in is 3 gallons per minute, whereas the mixed liquid is pumped out at a rate of 4 gallons per minute. Find the number of gallons of alcohol $A(t)$ in the tank at any time. What is the percentage of alcohol in the tank after 60 minutes? When is the tank empty?

Solution:

If $A(t)$ is the amount present at any time, then $\frac{dA}{dt}$ is that rate at which the amount of alcohol is changing.

$$\frac{dA}{dt} = \text{(rate of subs entering)} - \text{(rate of subs leaving)} = R_1 - R_2$$

$$R_1 = \left(\frac{6 \text{ gal alcohol}}{100 \text{ gal solution}}\right)\left(\frac{3 \text{ gal solution}}{1 \text{ min}}\right) = 0.18\frac{\text{gal}}{\text{min}}$$

$$R_2 = \left(\frac{A \text{ gal alcohol}}{400 - t \text{ gal solution}}\right)\left(\frac{4 \text{ gal solution}}{1 \text{ min}}\right) = \frac{4A}{400 - t}\frac{\text{gal}}{\text{min}}$$

$$\frac{dA}{dt} = 0.18 - \frac{4A}{400 - t}$$

$$\frac{dA}{dt} + \frac{4A}{400 - t} = 0.18 \tag{9}$$

Equation (9) is a linear differential equation.

$$\mu(t) = e^{-4(\ln(400-t))}$$

$$\mu(t) = (400 - t)^{-4}$$

$$d[(400 - t)^{-4}A] = 0.18(400 - t)^{-4}$$

$$(400 - t)^{-4}A = 0.06(400 - t)^{-3} + C$$

$$A = 0.06(400 - t) + C(400 - t)^4$$

We know that $A(0) = 12$ so:

$$12 = 0.06(400 - 0) + C(400 - 0)^4$$

$$-12 = 400^4 C$$

$$C = -4.6875 \times 10^{-10}$$

So the number of gallons of alcohol in the tank at any time $A(t)$ is:

$$A(t) = 0.06(400 - t) - 4.6875 \times 10^{-10}(400 - t)^4 \tag{10}$$

After 60 minutes, the percentage of alcohol is obtained by dividing the amount of alcohol remaining by the amount of solution in the tank:

$$A(60) = 0.06(400 - 60) - 4.6875 \times 10^{-10}(400 - 60)^4 \approx 14.135925 \text{ gal}$$

Every minute, 1 gal of solution disappears from the tank so after 60 minutes, $400 - 60 = 340$ gal remain.

$$\frac{14.135925}{340} \times 100 = 4.157625\%$$

The tank will be empty after 400 minutes because with every minute 1 gal of solution disappears.

Problem: (p 107, #34)

When forgetfulness is taken into account, the rate of memorization of a subject is given by

$$\frac{dA}{dt} = k_1(M - A) - k_2 A,$$

where $k_1 > 0$, $k_2 > 0$, $A(t)$ is the amount of material memorized in time t, M is the total amount to be memorized, and $M - A$ is the amount remaining to be memorized. Solve for $A(t)$ and graph the solution. Assume $A(0) = 0$. Find the limiting value of A as $t \to \infty$ and interpret the result.

Solution:

$$\frac{dA}{dt} = k_1(M - A) - k_2 A$$
$$\frac{dA}{dt} = k_1 M - k_1 A - k_2 A$$
$$\frac{dA}{dt} = k_1 M - A(k_1 + k_2)$$
$$\frac{dA}{k_1 M - A(k_1 + k_2)} = dt$$
$$\int \frac{dA}{k_1 M - A(k_1 + k_2)} = \int dt$$
$$\frac{\ln(k_1 M - A(k_1 + k_2))}{-(k_1 + k_2)} = t + C$$
$$\ln(k_1 M - A(k_1 + k_2)) = -t(k_1 + k_2) + C_1$$
$$(k_1 M - A(k_1 + k_2)) = C_2 e^{-t(k_1+k_2)}$$
$$-A(k_1 + k_2) = C_2 e^{-t(k_1+k_2)} - k_1 M$$
$$A = \frac{C_2 e^{-t(k_1+k_2)} - k_1 M}{-(k_1 + k_2)} \tag{11}$$

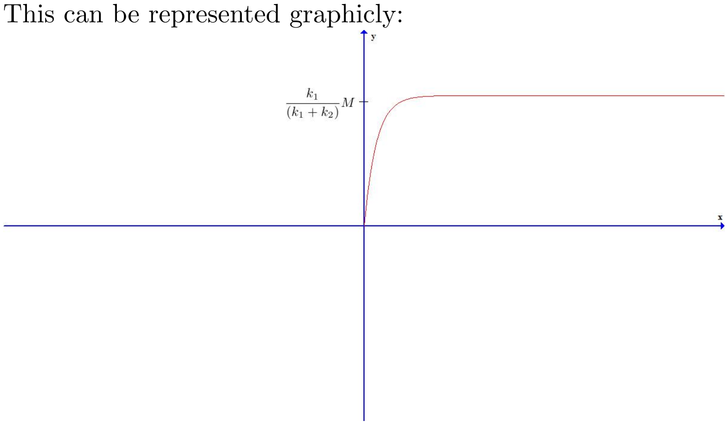
It is given that $A(0) = 0$. Plugging that into Equation (11) yields:

$$0 = \frac{C - k_1 M}{-(k_1 + k_2)}$$
$$C = k_1 M$$
$$A(t) = \frac{k_1 M(e^{-t(k_1+k_2)}) - 1}{-(k_1 + k_2)}$$

As time goes on,

$$\lim_{t\to\infty} \frac{k_1 M(e^{-t(k_1+k_2)} - 1)}{-(k_1 + k_2)} = \frac{k_1}{(k_1 + k_2)} M$$

This can be represented graphicly:



This means that relatively quickly, the amount memorized becomes constant. That constant is always less than the total amount to be memorized.

Problem: (p 116, #4)
Find the solution of the modified logistic equation

$$\frac{dP}{dt} = P(a - bP)(1 - cP^{-1}),\ \ a, b, c > 0$$

Solution:

$$\frac{dP}{P(a - bP)(1 - cP^{-1})} = dt$$
$$\int \frac{dP}{P(a - bP)(1 - cP^{-1})} = \int dt$$
$$\int \frac{dP}{(a - bP)(P - c)} = \int dt$$

Partial fractions yield:

$$\frac{1}{(a - bP)(1 - c)} = \frac{A}{a - bP} + \frac{B}{P - c}$$
$$1 = A(P - c) + B(a - bP)$$
$$1 = AP - Ac + Ba - bPB$$
$$1 = (-Ac + Ba) + (A - bB)P$$
$$\begin{cases} 1 = -Ac + Ba \\ 0 = A - bB \end{cases}$$
$$A = bB$$
$$1 = -(bB)c + Ba$$
$$1 = (-bc + a)B$$
$$B = \frac{1}{a - bc}$$
$$A = \frac{b}{a - bc}$$

$$\int \frac{dP}{(a - bP)(P - c)} = \int \left[\frac{b/(a - bc)}{a - bP} + \frac{1/(a - bc)}{P - c}\right] dP = \int dt$$
$$\frac{1}{a - bc} \int \left[\frac{b}{a - bP} + \frac{1}{P - c}\right] dP = \int dt$$
$$\frac{1}{a - bc}(-\ln(a - bP) + \ln(P - c)) = t + Q,\ \ Q \text{ is a constant}$$
$$\frac{\ln\left(\frac{P - c}{a - bP}\right)}{a - bc} = t + Q$$
$$\ln\left(\frac{P - c}{a - bP}\right) = (t + Q)(a - bc)$$
$$\frac{P - c}{a - bP} = e^{(t+Q)(a-bc)}$$
$$P - c = e^{(t+Q)(a-bc)}(a - bP)$$
$$P = c + e^{(t+Q)(a-bc)}a - bPe^{(t+Q)(a-bc)}$$
$$P + be^{(t+Q)(a-bc)}P = e^{(t+Q)(a-bc)}a + c$$
$$(1 + be^{(t+Q)(a-bc)})P = e^{(t+Q)(a-bc)}a + c$$
$$P = \frac{ae^{(t+Q)(a-bc)} + c}{1 + be^{(t+Q)(a-bc)}}$$

Problem: (p 117, #10)

In a third-order chemical reaction the number of grams X of a compound obtained by combining three chemicals is governed by

$$\frac{dX}{dt} = k(\alpha - X)(\beta - X)(\gamma - X).$$

Solve the equation under the assumption $\alpha \neq \beta \neq \gamma$.

Solution:

$$\frac{dX}{(\alpha - X)(\beta - X)(\gamma - X)} = k\,dt$$

$$\int \frac{dX}{(\alpha - X)(\beta - X)(\gamma - X)} = \int k\,dt$$

Partial fractions yield:

$$\frac{1}{(\alpha - X)(\beta - X)(\gamma - X)} = \frac{A}{\alpha - X} + \frac{B}{\beta - X} + \frac{C}{\gamma - X}$$

$$1 = A(\beta - X)(\gamma - X) + B(\alpha - X)(\gamma - X) + C(\beta - X)(\gamma - X)$$

$$1 = A(\beta\gamma - \gamma X - \beta X + X^2) + B(\alpha\gamma - \alpha X - \gamma X + X^2) + C(\beta\gamma - \beta X - \gamma X + X^2)$$

$$1 = A\beta\gamma - A\gamma X - A\beta X + AX^2 + B\alpha\gamma - B\alpha X - B\gamma X + BX^2 + C\beta\gamma - C\beta X - C\gamma X + CX^2$$

$$\begin{cases} 1 = A\beta\gamma + B\alpha\gamma + C\alpha\beta \\ 0 = -A\gamma - A\beta - B\alpha - B\gamma - C\alpha - C\beta \\ 0 = A + B + C \end{cases}$$

$$\begin{cases} 1 = A\beta\gamma + B\alpha\gamma + C\alpha\beta \\ 0 = -A(\gamma + \beta) - B(\alpha + \gamma) - C(\alpha + \beta) \\ 0 = A + B + C \end{cases}$$

$$A = -B - C$$

$$0 = -A(\gamma + \beta) - B(\alpha + \gamma) - C(\alpha + \beta)$$

$$0 = (B + C)(\gamma + \beta) - B(\alpha + \gamma) - C(\alpha + \beta)$$

$$0 = B(\gamma + \beta) + C(\gamma + \beta) - B(\alpha + \gamma) - C(\alpha + \beta)$$

$$0 = B(\gamma + \beta - \alpha - \gamma) + C(\gamma + \beta - \alpha - \beta)$$

$$0 = B(\beta - \alpha) + C(\gamma - \alpha)$$

$$B = \frac{-C(\gamma - \alpha)}{\beta - \alpha}$$

$$1 = A\beta\gamma + B\alpha\gamma + C\alpha\beta$$

$$1 = (-B - C)\beta\gamma + B\alpha\gamma + C\alpha\beta$$

$$1 = -B\beta\gamma - C\beta\gamma + B\alpha\gamma + C\alpha\beta$$

$$1 = B(\alpha\gamma - \beta\gamma) + C(\alpha\beta - \beta\gamma)$$

$$1 = B\gamma(\alpha - \beta) + C\beta(\alpha - \gamma)$$

$$1 = -B\gamma(\beta - \alpha) - C\beta(\gamma - \alpha)$$

$$1 = -\frac{-C(\gamma - \alpha)}{\beta - \alpha}\gamma(\beta - \alpha) - C\beta(\gamma - \alpha)$$

$$1 = C\gamma(\gamma-\alpha) - C\beta(\gamma-\alpha)$$
$$1 = C(\gamma-\alpha)(\gamma-\beta)$$
$$C = \frac{1}{(\gamma-\alpha)(\gamma-\beta)}$$

$$B = \frac{-C(\gamma-\alpha)}{\beta-\alpha}$$

$$B = \frac{-(\gamma-\alpha)}{(\beta-\alpha)(\gamma-\alpha)(\gamma-\beta)}$$
$$B = \frac{-1}{(\beta-\alpha)(\gamma-\beta)}$$

$$A = -B - C$$
$$A = -\frac{-1}{(\beta-\alpha)(\gamma-\beta)} - \frac{1}{(\gamma-\alpha)(\gamma-\beta)}$$
$$A = \frac{(\gamma-\alpha)-(\beta-\alpha)}{(\beta-\alpha)(\gamma-\beta)(\gamma-\alpha)}$$
$$A = \frac{1}{(\beta-\alpha)(\gamma-\alpha)}$$

Now back to the differential equation:

$$\int \frac{dX}{(\alpha-X)(\beta-X)(\gamma-X)} = \int k\,dt$$
$$\int \left[\frac{\frac{1}{(\beta-\alpha)(\gamma-\alpha)}}{\alpha-X} - \frac{\frac{1}{(\beta-\alpha)(\gamma-\beta)}}{\beta-X} + \frac{\frac{1}{(\gamma-\alpha)(\gamma-\beta)}}{\gamma-X} \right] dX = \int k\,dt$$
$$\frac{-\ln(\alpha-X)}{(\beta-\alpha)(\gamma-\alpha)} + \frac{\ln(\beta-X)}{(\beta-\alpha)(\gamma-\beta)} - \frac{\ln(\gamma-X)}{(\gamma-\alpha)(\gamma-\beta)} = kt + C$$

BONUS Problem: (p 118, #14)

According to Stefan's law of radiation, the rate of change of temperature from a body at absolute temperature T is

$$\frac{dT}{dt} = k(T^4 - T_m^4),$$

where T_m is the absolute temperature of the surrounding medium. Find a solution of this differential equation. It can be shown that when $T - T_m$ is small compared to T_m, this particular equation is closely approximated by Newton's law of cooling,

$$\frac{dT}{dt} = k(T - T_m)$$

Solution:

$$\frac{dT}{dt} = k(T^4 - T_m^4)$$

$$\frac{dT}{dt} = k(T - T_m)(T + T_m)(T^2 + T_m^2)$$

$$\frac{dT}{(T - T_m)(T + T_m)(T^2 + T_m^2)} = k\,dt$$

$$\int \frac{dT}{(T - T_m)(T + T_m)(T^2 + T_m^2)} = \int k\,dt$$

Partial fractions yield:

$$\frac{1}{(T - T_m)(T + T_m)(T^2 + T_m^2)} = \frac{A}{T - T_m} + \frac{B}{T + T_m} + \frac{CT + D}{T^2 + T_m^2}$$

$$1 = A(T + T_m)(T^2 + T_m^2) + B(T - T_m)(T^2 + T_m^2) + (CT + D)(T - T_m)(T + T_m)$$

$$1 = A(T^3 + T_mT^2 + T_m^2T + T_m^3) + B(T^3 - T_mT^2 + T_m^2T - T_m^3) + (CT + D)(T^2 - T_m^2)$$

$$1 = AT^3 + AT_mT^2 + AT_m^2T + AT_m^3 + BT^3 - BT_mT^2 + BT_m^2T - BT_m^3 + CT^3 + DT^2 - CTT_m^2 - DT_m^2$$

$$1 = (A + B + C)T^3 + (AT_m - BT_m + D)T^2 + (AT_m^2 + BT_m^2 - CT_m^2)T + (AT_m^3 - BT_m^3 - DT_m^2)$$

$$\begin{cases} 0 = A + B + C \\ 0 = AT_m - BT_m + D \\ 0 = AT_m^2 + BT_m^2 - CT_m^2 \\ 1 = AT_m^3 - BT_m^3 - DT_m^2 \end{cases}$$

$$A = -B - C$$

$$0 = (-B - C)T_m^2 + BT_m^2 - CT_m^2$$

$$0 = -BT_m^2 - CT_m^2 + BT_m^2 - CT_m^2$$

$$0 = -2CT_m^2$$

$$C = 0$$

$$A = -B$$

$$0 = AT_m - BT_m + D$$

$$0 = -BT_m - BT_m + D$$

$$0 = -2BT_m + D$$

$$D = 2BT_m$$

$$\begin{aligned} 1 &= AT_m^3 - BT_m^3 - DT_m^2 \\ 1 &= -BT_m^3 - BT_m^3 - 2BT_mT_m^2 \\ 1 &= -2BT_m^3 - 2BT_m^3 \\ 1 &= -4BT_m^3 \\ B &= \frac{-1}{4T_m^3} \end{aligned}$$

$$\begin{aligned} D &= 2BT_m \\ D &= 2\frac{-1}{4T_m^3}T_m \\ D &= \frac{-1}{2T_m^2} \end{aligned}$$

$$\begin{aligned} A &= -B \\ A &= \frac{1}{4T_m^3} \end{aligned}$$

Now back to the differential equation:

$$\int \frac{dT}{(T-T_m)(T+T_m)(T^2+T_m^2)} = \int k\,dt$$

$$\int \left(\frac{\frac{1}{4T_m^3}}{T-T_m} + \frac{\frac{-1}{4T_m^3}}{T+T_m} + \frac{\frac{-1}{2T_m^2}}{T^2+T_m^2} \right) dT = \int k\,dt$$

$$\frac{\ln(T-T_m)}{4T_m^3} - \frac{\ln(T+T_m)}{4T_m^3} - \frac{1}{2T_m^2}\int \frac{dT}{T^2-T_m^2} = kt + C \tag{12}$$

In equation (12),

$$\int \frac{dT}{T^2-T_m^2}$$

$$\text{Let } T_m\tan(\theta) = T$$

$$\text{and } dT = T_m\sec^2(\theta)\,d\theta$$

$$\int \frac{dT}{T^2-T_m^2} = \int \frac{T_m\sec^2(\theta)}{(T_m\tan(\theta))^2+T_m^2}\,d\theta =$$

$$\int \frac{T_m\sec^2(\theta)}{T_m^2\tan^2(\theta)+T_m^2}\,d\theta =$$

$$\int \frac{T_m\sec^2(\theta)}{T_m^2(\tan^2(\theta)+1)}\,d\theta =$$

$$\int \frac{\sec^2(\theta)}{T_m\sec^2(\theta)}\,d\theta =$$

$$\int \frac{d\theta}{T_m} =$$

$$\frac{1}{T_m}\tan^{-1}\left(\frac{T}{T_m}\right)$$

$$\begin{aligned} \frac{\ln(T-T_m)}{4T_m^3} - \frac{\ln(T+T_m)}{4T_m^3} - \frac{1}{2T_m^2}\int \frac{dT}{T^2-T_m^2} &= kt + C \\ \frac{\ln(T-T_m)}{4T_m^3} - \frac{\ln(T+T_m)}{4T_m^3} - \frac{1}{2T_m^2}\frac{1}{T_m}\tan^{-1}\left(\frac{T}{T_m}\right) &= kt + C \\ \frac{\ln(T-T_m)}{4T_m^3} - \frac{\ln(T+T_m)}{4T_m^3} - \frac{1}{2T_m^3}\tan^{-1}\left(\frac{T}{T_m}\right) &= kt + C \end{aligned}$$