Linear Algebra Writing Prompt 4

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Theorem 4.2, Axiom 3
Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\Re^n$, and let c and d be scalars.

$$(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$$

Proof: Let $\mathbf{u}=(u_1,u_2,\ldots,u_n)$, $\mathbf{v}=(v_1,v_2,\ldots,v_n)$, and $\mathbf{w}=(w_1,w_2,\ldots,w_n)$ be vectors in $\Re^n$. Note that u_i, v_i, and w_i are real numbers for all i. Consider

$$\begin{aligned}
(\mathbf{u}+\mathbf{v})+\mathbf{w} &= ((u_1,u_2,\ldots,u_n)+(v_1,v_2,\ldots,v_n))+(w_1,w_2,\ldots,w_n) \\
&= ((u_1+v_1),(u_2+v_2),\ldots,(u_n+v_n))+(w_1,w_2,\ldots,w_n) && \text{(Definition of Addition of Vectors)} \\
&= ((u_1+v_1)+w_1,(u_2+v_2)+w_2,\ldots,(u_n+v_n)+w_n) && \text{(Definition of Addition of Vectors)} \\
&= (u_1+(v_1+w_1),u_2+(v_2+w_2),\ldots,u_n+(v_n+w_n)) && \text{(Associative Property of Real Numbers)} \\
&= (u_1,u_2,\ldots,u_n)+((v_1+w_1),(v_2+w_2),\ldots,(v_n+w_n)) && \text{(Definition of Addition of Vectors)} \\
&= (u_1,u_2,\ldots,u_n)+((v_1,v_2,\ldots,v_n)+(w_1,w_2,\ldots,w_n)) && \text{(Definition of Addition of Vectors)} \\
&= \mathbf{u}+(\mathbf{v}+\mathbf{w})
\end{aligned}$$

QED

Theorem 4.2, Axiom 4
Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\Re^n$, and let c and d be scalars.

$$\mathbf{u}+\mathbf{0}=\mathbf{u}$$

Proof: Let $\mathbf{u}=(u_1,u_2,\ldots,u_n)$ be a vector in $\Re^n$. Note that u_i is a real number for all i. Consider

$$\begin{aligned}
\mathbf{u}+\mathbf{0} &= (u_1,u_2,\ldots,u_n)+(0,0,\ldots,0) \\
&= (u_1+0,u_2+0,\ldots,u_n+0) && \text{(Definition of Addition of Vectors)} \\
&= (u_1,u_2,\ldots,u_n) && \text{(Since 0 is the Additive Identity of Real Numbers)} \\
&= \mathbf{u}
\end{aligned}$$

QED

Theorem 4.2, Axiom 7
Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\Re^n$, and let c and d be scalars.

$$c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$$

Proof: Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be vectors in $\Re^n$. Let c be a scalar. Note that u_i and v_i are real numbers for all i. Consider

$$\begin{aligned} c(\mathbf{u}+\mathbf{v}) &= c\left((u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n)\right) \\ &= c\left((u_1+v_1), (u_2+v_2), \ldots, (u_n+v_n)\right) && \text{(Definition of Addition of Vectors)} \\ &= (c(u_1+v_1), c(u_2+v_2), \ldots, c(u_n+v_n)) && \text{(Definition of Scalar Multiplication)} \\ &= (cu_1+cv_1, cu_2+cv_2, \ldots, cu_n+cv_n) && \text{(Distributive Property of Real Numbers)} \\ &= (cu_1, cu_2, \ldots, cu_n) + (cv_1, cv_2, \ldots, cv_n) && \text{(Definition of Addition of Vectors)} \\ &= c(u_1, u_2, \ldots, u_n) + c(v_1, v_2, \ldots, v_n) && \text{(Definition of Scalar Multiplication)} \\ &= c\mathbf{u} + c\mathbf{v} \end{aligned}$$

QED

Theorem 4.3, Axiom 3
Let $\mathbf{v}$ be a vector in $\Re^n$, and let c be a scalar.

$$0\mathbf{v} = \mathbf{0}$$

Proof: Let $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be a vector in $\Re^n$ and let c be a scalar. Note that v_i is a real number for all i. Consider

$$\begin{aligned} 0\mathbf{v} &= 0(v_1, v_2, \ldots, v_n) \\ &= (0v_1, 0v_2, \ldots, 0v_n) && \text{(Definition of Scalar Multiplication)} \\ &= (0, 0, \ldots, 0) && \text{(Since 0 multiplied by any real number is 0)} \\ &= \mathbf{0} \end{aligned}$$

QED

Theorem 4.3, Axiom 5
Let $\mathbf{v}$ be a vector in $\Re^n$, and let c be a scalar.

$$\text{If } c\mathbf{v} = \mathbf{0}, \text{ then } c = 0 \text{ or } \mathbf{v} = \mathbf{0}$$

Proof: Let $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be a vector in $\Re^n$ and let c be a scalar such that $c\mathbf{v} = \mathbf{0}$. Note that v_i is a real number for all i. From the definition of scalar multiplication $c\mathbf{v} = c(v_1, v_2, \ldots, v_n) = (cv_1, cv_2, \ldots, cv_n)$. Since $c\mathbf{v} = \mathbf{0}$, $(cv_1, cv_2, \ldots, cv_n) = (0, 0, \ldots, 0)$. This implies that $cv_1 = 0$, $cv_2 = 0$, …, $cv_n = 0$. By the zero factor property of real numbers, $cv_i = 0$ implies that $c = 0$ or $v_i = 0$ for all i. If $c = 0$ we are done. If $c \neq 0$ then $v_i = 0$ for all i or $(v_1, v_2, \ldots v_n) = (0, 0, \ldots, 0)$ which means $\mathbf{v} = \mathbf{0}$. QED

Theorem 4.3, Axiom 6
Let $\mathbf{v}$ be a vector in $\Re^n$, and let c be a scalar.

$$-(-\mathbf{v}) = \mathbf{v}$$

Proof: Let $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be a vector in $\Re^n$. Note that v_i is a real number for all i. Consider

$$\begin{aligned}
-(-\mathbf{v}) &= -1(-1(v_1, v_2, \ldots, v_n)) \\
&= -1(-1v_1, -1v_2, \ldots, -1v_n) && \text{(Definition of Scalar Multiplication)} \\
&= (-1(-1v_1), -1(-1v_2), \ldots, -1(-1v_n)) && \text{(Definition of Scalar Multiplication)} \\
&= ((-1 \cdot -1)v_1, (-1 \cdot -1)v_2, \ldots, (-1 \cdot -1)v_n) && \text{(Associative Property of Real Numbers)} \\
&= (1v_1, 1v_2, \ldots, 1v_n) && (-1 \cdot -1 = 1) \\
&= (v_1, v_2, \ldots, v_n) && \text{(Since 1 is the Multiplicative Identity of Real Numbers)} \\
&= \mathbf{v}
\end{aligned}$$

QED

Sec 4.2, 27
Prove in full detail that $M_{2,2}$, with the standard operations, is a vector space.

Proof: Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, and $C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$ be elements of $M_{2,2}$. Let d and e be scalars. Note that a_{ij}, b_{ij}, and c_{ij} are real numbers.

1. $A + B$ is in $M_{2,2}$.

$$\begin{aligned}
A + B &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\
&= \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix} && \text{(Definition of Addition of Matrices)}
\end{aligned}$$

Since the real numbers are closed under addition, $a_{ij} + b_{ij}$ is a real number. Therefore $\begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}$ is an element of $M_{2,2}$.

2. $A + B = B + A$.

$$\begin{aligned}
A + B &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\
&= \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix} && \text{(Definition of Addition of Matrices)}
\end{aligned}$$

$$\begin{aligned}
B + A &= \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\
&= \begin{pmatrix} b_{11} + a_{11} & b_{12} + a_{12} \\ b_{21} + a_{21} & b_{22} + a_{22} \end{pmatrix} && \text{(Definition of Addition of Matrices)} \\
&= \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix} && \text{(Commutative Property of Real Numbers)}
\end{aligned}$$

Therefore $A + B = B + A$.

3. $A + (B + C) = (A + B) + C$.

$$\begin{aligned}
A + (B + C) &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \left(\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} + \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \right) \\
&= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} + c_{11} & b_{12} + c_{12} \\ b_{21} + c_{21} & b_{22} + c_{22} \end{pmatrix} && \text{(Definition of Addition of Matrices)} \\
&= \begin{pmatrix} a_{11} + (b_{11} + c_{11}) & a_{12} + (b_{12} + c_{12}) \\ a_{21} + (b_{21} + c_{21}) & a_{22} + (b_{22} + c_{22}) \end{pmatrix} && \text{(Definition of Addition of Matrices)}
\end{aligned}$$

$$
\begin{aligned}
(A+B)+C &= \left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right) + \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \\
&= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix} + \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} && \text{(Definition of Addition of Matrices)} \\
&= \begin{pmatrix} (a_{11}+b_{11})+c_{11} & (a_{12}+b_{12})+c_{12} \\ (a_{21}+b_{21})+c_{21} & (a_{22}+b_{22})+c_{22} \end{pmatrix} && \text{(Definition of Addition of Matrices)} \\
&= \begin{pmatrix} a_{11}+(b_{11}+c_{11}) & a_{12}+(b_{12}+c_{12}) \\ a_{21}+(b_{21}+c_{21}) & a_{22}+(b_{22}+c_{22}) \end{pmatrix} && \text{(Associative Property of Real Numbers)}
\end{aligned}
$$

Therefore $A+(B+C)=(A+B)+C$.

4. $M_{2,2}$ has a zero vector $\mathbf{0}$ such that for every A in $M_{2,2}$, $A+\mathbf{0}=A$.

   Define the zero vector to be $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

$$
\begin{aligned}
A+\mathbf{0} &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} a_{11}+0 & a_{12}+0 \\ a_{21}+0 & a_{22}+0 \end{pmatrix} && \text{(Definition of Matrix Addition)} \\
&= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} && \text{(Since 0 is the Additive Identity of Real Numbers)} \\
&= A
\end{aligned}
$$

5. For every A in $M_{2,2}$, there is a vector in $M_{2,2}$ denoted by $-A$ such that $A+(-A)=\mathbf{0}$.

   Define the additive inverse to be $-A = \begin{pmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{pmatrix}$.

$$
\begin{aligned}
A+(-A) &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{pmatrix} \\
&= \begin{pmatrix} a_{11}+(-a_{11}) & a_{12}+(-a_{12}) \\ a_{21}+(-a_{21}) & a_{22}+(-a_{22}) \end{pmatrix} && \text{(Definition of Addition of Matrices)} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} && \text{(Additive Inverses of Real Numbers)} \\
&= \mathbf{0}
\end{aligned}
$$

6. dA is in $M_{2,2}$.

$$
\begin{aligned}
dA &= d\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\
&= \begin{pmatrix} da_{11} & da_{12} \\ da_{21} & da_{22} \end{pmatrix} && \text{(Definition of Scalar Multiplication)}
\end{aligned}
$$

   Since the real numbers are closed under multiplication, da_{ij} is a real number. Therefore $\begin{pmatrix} da_{11} & da_{12} \\ da_{21} & da_{22} \end{pmatrix}$ is an element of $M_{2,2}$.

7. $d(A+B)=dA+dB$.

$$
\begin{aligned}
d(A+B) &= d\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right) \\
&= d\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix} && \text{(Definition of Addition of Matrices)}
\end{aligned}
$$

$$= \begin{pmatrix} d(a_{11}+b_{11}) & d(a_{12}+b_{12}) \\ d(a_{21}+b_{21}) & d(a_{22}+b_{22}) \end{pmatrix} \quad \text{(Definition of Scalar Multiplication)}$$

$$= \begin{pmatrix} da_{11}+db_{11} & da_{12}+db_{12} \\ da_{21}+db_{21} & da_{22}+db_{22} \end{pmatrix} \quad \text{(Distributive Property of Real Numbers)}$$

$$dA + dB = d\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + d\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$= \begin{pmatrix} da_{11} & da_{12} \\ da_{21} & da_{22} \end{pmatrix} + \begin{pmatrix} db_{11} & db_{12} \\ db_{21} & db_{22} \end{pmatrix} \quad \text{(Definition of Scalar Multiplication)}$$

$$= \begin{pmatrix} da_{11}+db_{11} & da_{12}+db_{12} \\ da_{21}+db_{21} & da_{22}+db_{22} \end{pmatrix} \quad \text{(Definition of Addition of Matrices)}$$

Therefore $d(A+B) = dA + dB$.

8. $(d+e)A = dA + eA$.

$$(d+e)A = (d+e)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$= \begin{pmatrix} (d+e)a_{11} & (d+e)a_{12} \\ (d+e)a_{21} & (d+e)a_{22} \end{pmatrix} \quad \text{(Definition of Scalar Multiplication)}$$

$$= \begin{pmatrix} da_{11}+ea_{11} & da_{12}+ea_{12} \\ da_{21}+ea_{21} & da_{22}+ea_{22} \end{pmatrix} \quad \text{(Distributive Property of Real Numbers)}$$

$$dA + eA = d\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + e\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$= \begin{pmatrix} da_{11} & da_{12} \\ da_{21} & da_{22} \end{pmatrix} + \begin{pmatrix} ea_{11} & ea_{12} \\ ea_{21} & ea_{22} \end{pmatrix} \quad \text{(Definition of Scalar Multiplication)}$$

$$= \begin{pmatrix} da_{11}+ea_{11} & da_{12}+ea_{12} \\ da_{21}+ea_{21} & da_{22}+ea_{22} \end{pmatrix} \quad \text{(Definition of Addition of Matrices)}$$

Therefore $(d+e)A = dA + eA$.

9. $d(eA) = (de)A$.

$$d(eA) = d\left(e\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right)$$

$$= d\begin{pmatrix} ea_{11} & ea_{12} \\ ea_{21} & ea_{22} \end{pmatrix} \quad \text{(Definition of Scalar Multiplication)}$$

$$= \begin{pmatrix} d(ea_{11}) & d(ea_{12}) \\ d(ea_{21}) & d(ea_{22}) \end{pmatrix} \quad \text{(Definition of Scalar Multiplication)}$$

$$(de)A = (de)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$= \begin{pmatrix} (de)a_{11} & (de)a_{12} \\ (de)a_{21} & (de)a_{22} \end{pmatrix} \quad \text{(Definition of Scalar Multiplication)}$$

$$= \begin{pmatrix} d(ea_{11}) & d(ea_{12}) \\ d(ea_{21}) & d(ea_{22}) \end{pmatrix} \quad \text{(Associative Property of Real Numbers)}$$

Therefore $d(eA) = (de)A$.

10. $1(A) = A$.

$$\begin{aligned} 1(A) &= 1 \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\ &= \begin{pmatrix} 1a_{11} & 1a_{12} \\ 1a_{21} & 1a_{22} \end{pmatrix} && \text{(Definition of Scalar Multiplication)} \\ &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} && \text{(Since 1 is the Multiplicative Identity of Real Numbers)} \\ &= A \end{aligned}$$

QED

Sec 4.2, 30
Let V be the set of all positive real numbers. Determine whether V is a vector space with the following operations.

$$\begin{aligned} x + y &= xy && \text{Addition} \\ cx &= x^c && \text{Scalar Multiplication} \end{aligned}$$

If it is, verify each vector space axiom: if not, state all vector space axioms that fail.

Answer: Yes, it is a vector space.
Proof: Let V be the set of all positive real numbers. Let x, y, and z be elements of V. Let c and d be scalars.

1. $x + y$ is in V.

$$x + y = xy$$

Since the real numbers are closed under multiplication, xy is a real numbers. Since x and y are positive numbers, xy is also positive. Therefore xy is an element of V.

2. $x + y = y + x$.

$$x + y = xy \qquad \text{(Definition of Addition)}$$

$$\begin{aligned} y + x &= yx && \text{(Definition of Addition)} \\ &= xy && \text{(Commutative Property of Multiplication of Real Numbers)} \end{aligned}$$

Therefore $x + y = y + x$.

3. $x + (y + z) = (x + y) + z$.

$$\begin{aligned} x + (y + z) &= x + yz && \text{(Definition of Addition)} \\ &= x(yz) && \text{(Definition of Addition)} \end{aligned}$$

$$\begin{aligned} (x + y) + z &= xy + z && \text{(Definition of Addition)} \\ &= (xy)z && \text{(Definition of Addition)} \\ &= x(yz) && \text{(Associative Property of Real Numbers)} \end{aligned}$$

Therefore $x + (y + z) = (x + y) + z$.

4. V has a zero vector $\mathbf{0}$ such that for every x in V, $x + \mathbf{0} = x$.
   Define the zero vector to be $\mathbf{0} = 1$.

$$\begin{aligned} x + \mathbf{0} &= x + 1 \\ &= x \cdot 1 && \text{(Definition of Addition)} \\ &= x && \text{(Since 1 is the Multiplicative Identity of Real Numbers)} \end{aligned}$$

5. For every x in V, there is a vector in V denoted by $-x$ such that $x + (-x) = \mathbf{0}$.
   Define the additive inverse to be $-x = \frac{1}{x}$.

$$\begin{aligned} x + (-x) &= x + \frac{1}{x} \\ &= x\frac{1}{x} && \text{(Definition of Addition)} \\ & && \text{(Note that } \frac{1}{x} \text{ is defined since } x \text{ is always positive and thus never 0)} \\ &= 1 && \text{(Multiplicative Inverses of Real Numbers)} \\ &= \mathbf{0} \end{aligned}$$

6. cx is in V.

$$cx = x^c \text{(Definition of Scalar Multiplication)}$$

   A positive real number raised to a real power is always a positive real number so x^c is an element of V.

7. $c(x + y) = cx + cy$.

$$\begin{aligned} c(x + y) &= c(xy) && \text{(Definition of Addition)} \\ &= (xy)^c && \text{(Definition of Scalar Multiplication)} \\ &= x^c y^c && \text{(Product to a Power Property)} \end{aligned}$$

$$\begin{aligned} cx + cy &= x^c + y^c && \text{(Definition of Scalar Multiplication)} \\ &= x^c y^c && \text{(Definition of Addition)} \end{aligned}$$

   Therefore $c(x + y) = cx + cy$.

8. $(c + d)x = cx + dx$.

$$\begin{aligned} (c + d)x &= x^{c+d} && \text{(Definition of Scalar Multiplication)} \\ &= x^c x^d && \text{(Product of Like Bases Property)} \end{aligned}$$

$$\begin{aligned} cx + dx &= x^c + x^d && \text{(Definition of Scalar Multiplication)} \\ &= x^c x^d && \text{(Definition of Addition)} \end{aligned}$$

   Therefore $(c + d)x = cx + dx$.

9. $c(dx) = (cd)x$.

$$\begin{aligned} c(dx) &= cx^d && \text{(Definition of Scalar Multiplication)} \\ &= \left(x^d\right)^c && \text{(Definition of Scalar Multiplication)} \\ &= x^{dc} && \text{(Power to a Power Property)} \\ &= x^{cd} && \text{(Commutative Property Property of Real Numbers)} \end{aligned}$$

$$(cd)x = x^{cd} \qquad \text{(Definition of Scalar Multiplication)}$$

   Therefore $c(dx) = (cd)x$.

10. $1(x) = x$.

$$\begin{aligned} 1(x) &= x^1 && \text{(Definition of Scalar Multiplication)} \\ &= x && \text{(Since any number to the power 1 is itself)} \end{aligned}$$

QED

Sec 4.2, 32
Prove that in a given vector space V, the additive inverse of a vector is unique.

Proof: Let V be a vector space. Let $\mathbf{u}$ be a vector in V. Assume that vector $\mathbf{u}$ has two additive inverses, $\mathbf{v}$ and $\mathbf{w}$. From the definition of an additive inverse we obtain

$$\mathbf{u} + \mathbf{v} = \mathbf{0}$$
$$\mathbf{u} + \mathbf{w} = \mathbf{0}.$$

Since $\mathbf{u} + \mathbf{v}$ and $\mathbf{v} + \mathbf{w}$ are both equal to $\mathbf{0}$ they are equal to each other:

$$\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}.$$

Adding $\mathbf{v}$ to the left side of each yields

$$\mathbf{v} + \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} + \mathbf{w}.$$

Since $\mathbf{v} + \mathbf{u} = \mathbf{0}$, the above expression becomes

$$\mathbf{0} + \mathbf{v} = \mathbf{0} + \mathbf{w}.$$

By VS property 4,

$$\mathbf{v} = \mathbf{w}.$$

This contradicts our assumption that $\mathbf{v} \neq \mathbf{w}$. Thus the additive inverse of a vector is unique. QED