$$\int \frac{dx}{x^4+1}$$

by Rustem Bilyalov

First we wish to factor the denominator in order to apply the partial fractions method. Thus we must complete the square in the denominator:

$$\begin{aligned} x^4+1 &= x^4+2x^2-2x^2+1 \\ &= (x^4+2x^2+1)-2x^2 \\ &= (x^2+1)^2-(\sqrt{2}x)^2. \end{aligned} \tag{1}$$

Noticing that (1) is a difference of two squares, we can rewrite it as

$$\left(x^2+\sqrt{2}x+1\right)\left(x^2-\sqrt{2}x+1\right). \tag{2}$$

Using (2) in place of the original denominator yields

$$\int \frac{dx}{\left(x^2+\sqrt{2}x+1\right)\left(x^2-\sqrt{2}x+1\right)}, \tag{3}$$

to which we can apply the partial fractions method. The partial fractions method involves finding A, B, C, and D such that

$$\frac{1}{\left(x^2+\sqrt{2}x+1\right)\left(x^2-\sqrt{2}x+1\right)} = \frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1}.$$

For this we will first find a common denominator for the two fractions on the right hand side and add them.

$$\frac{1}{\left(x^2+\sqrt{2}x+1\right)\left(x^2-\sqrt{2}x+1\right)} = \frac{(Ax+B)\left(x^2-\sqrt{2}x+1\right)+(Cx+D)\left(x^2+\sqrt{2}x+1\right)}{\left(x^2+\sqrt{2}x+1\right)\left(x^2-\sqrt{2}x+1\right)}. \tag{4}$$

Multiplying (4) through by the common denominator yields

$$1 = (Ax+B)\left(x^2-\sqrt{2}x+1\right)+(Cx+D)\left(x^2+\sqrt{2}x+1\right). \tag{5}$$

Multiplying out the right hand side in (5) results in

$$1 = Ax^3 - A\sqrt{2}x^2 + Ax + Bx^2 - B\sqrt{2}x + B + Cx^3 + C\sqrt{2}x^2 + Cx + Dx^2 + D\sqrt{2}x + D. \tag{6}$$

Grouping the coefficients of x^3, x^2, x, and 1 in (6) yields

$$0x^3+0x^2+0x+1 = (A+C)x^3 + (-A\sqrt{2}+B+C\sqrt{2}+D)x^2 + (A-B\sqrt{2}+C+D\sqrt{2})x + (B+D). \tag{7}$$

From (7) we obtain a system of 4 linear equations:

$$\begin{cases} A + C = 0 \\ -\sqrt{2}A + B + \sqrt{2}C + D = 0 \\ A - \sqrt{2}B + C + \sqrt{2}D = 0 \\ B + D = 1. \end{cases}$$

Proceeding to solve this system using Cramer's rule results in

$$A = \frac{\begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & \sqrt{2} & 1 \\ 0 & -\sqrt{2} & 1 & \sqrt{2} \\ 1 & 1 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 1 & 0 \\ -\sqrt{2} & 1 & \sqrt{2} & 1 \\ 1 & -\sqrt{2} & 1 & \sqrt{2} \\ 0 & 1 & 0 & 1 \end{vmatrix}} = \frac{1\begin{vmatrix} 0 & 1 & 1 \\ 0 & -\sqrt{2} & \sqrt{2} \\ 1 & 1 & 1 \end{vmatrix}}{1\begin{vmatrix} 1 & \sqrt{2} & 1 \\ -\sqrt{2} & 1 & \sqrt{2} \\ 1 & 0 & 1 \end{vmatrix} + 1\begin{vmatrix} -\sqrt{2} & 1 & 1 \\ 1 & -\sqrt{2} & \sqrt{2} \\ 0 & 1 & 1 \end{vmatrix}}$$

$$= \frac{\sqrt{2}-(-\sqrt{2})}{\sqrt{2}\sqrt{2}-1+1-(-\sqrt{2}\sqrt{2})+(-\sqrt{2}(-\sqrt{2}-\sqrt{2}))} = \frac{2\sqrt{2}}{2+2+4} = \frac{2\sqrt{2}}{8} = \frac{1}{2\sqrt{2}},$$

$$B = \frac{\begin{vmatrix} 1 & 0 & 1 & 0 \\ -\sqrt{2} & 0 & \sqrt{2} & 1 \\ 1 & 0 & 1 & \sqrt{2} \\ 0 & 1 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 1 & 0 \\ -\sqrt{2} & 1 & \sqrt{2} & 1 \\ 1 & -\sqrt{2} & 1 & \sqrt{2} \\ 0 & 1 & 0 & 1 \end{vmatrix}} = \frac{1\begin{vmatrix} 1 & 1 & 0 \\ -\sqrt{2} & \sqrt{2} & 1 \\ 1 & 1 & \sqrt{2} \end{vmatrix}}{1\begin{vmatrix} 1 & \sqrt{2} & 1 \\ -\sqrt{2} & 1 & \sqrt{2} \\ 1 & 0 & 1 \end{vmatrix} + 1\begin{vmatrix} -\sqrt{2} & 1 & 1 \\ 1 & -\sqrt{2} & \sqrt{2} \\ 0 & 1 & 1 \end{vmatrix}}$$

$$= \frac{\sqrt{2}\sqrt{2}-1-1(-\sqrt{2}\sqrt{2}-1)}{\sqrt{2}\sqrt{2}-1+1-(-\sqrt{2}\sqrt{2})+(-\sqrt{2}(-\sqrt{2}-\sqrt{2}))} = \frac{2-1+3}{2+2+4} = \frac{1}{2},$$

$$C = \frac{\begin{vmatrix} 1 & 0 & 0 & 0 \\ -\sqrt{2} & 1 & 0 & 1 \\ 1 & -\sqrt{2} & 0 & \sqrt{2} \\ 0 & 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 1 & 0 \\ -\sqrt{2} & 1 & \sqrt{2} & 1 \\ 1 & -\sqrt{2} & 1 & \sqrt{2} \\ 0 & 1 & 0 & 1 \end{vmatrix}} = \frac{-1\begin{vmatrix} 1 & 0 & 0 \\ -\sqrt{2} & 1 & 1 \\ 1 & -\sqrt{2} & \sqrt{2} \end{vmatrix}}{1\begin{vmatrix} 1 & \sqrt{2} & 1 \\ -\sqrt{2} & 1 & \sqrt{2} \\ 1 & 0 & 1 \end{vmatrix} + 1\begin{vmatrix} -\sqrt{2} & 1 & 1 \\ 1 & -\sqrt{2} & \sqrt{2} \\ 0 & 1 & 1 \end{vmatrix}}$$

$$= \frac{-1(\sqrt{2}-(-\sqrt{2}))}{\sqrt{2}\sqrt{2}-1+1-(-\sqrt{2}\sqrt{2})+(-\sqrt{2}(-\sqrt{2}-\sqrt{2}))} = \frac{-2\sqrt{2}}{2+2+4} = \frac{-2\sqrt{2}}{8} = \frac{-1}{2\sqrt{2}},$$

and

$$D = \frac{\begin{vmatrix} 1 & 0 & 1 & 0 \\ -\sqrt{2} & 1 & \sqrt{2} & 0 \\ 1 & -\sqrt{2} & 1 & 0 \\ 0 & 1 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 1 & 0 \\ -\sqrt{2} & 1 & \sqrt{2} & 1 \\ 1 & -\sqrt{2} & 1 & \sqrt{2} \\ 0 & 1 & 0 & 1 \end{vmatrix}} = \frac{1\begin{vmatrix} 1 & 0 & 1 \\ -\sqrt{2} & 1 & \sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{vmatrix}}{1\begin{vmatrix} 1 & \sqrt{2} & 1 \\ -\sqrt{2} & 1 & \sqrt{2} \\ 1 & 0 & 1 \end{vmatrix} + 1\begin{vmatrix} -\sqrt{2} & 1 & 1 \\ 1 & -\sqrt{2} & \sqrt{2} \\ 0 & 1 & 1 \end{vmatrix}}$$

$$= \frac{1-\sqrt{2}(-\sqrt{2})+(-\sqrt{2}(-\sqrt{2}))-1}{\sqrt{2}\sqrt{2}-1+1-(-\sqrt{2}\sqrt{2})+(-\sqrt{2}(-\sqrt{2}-\sqrt{2}))} = \frac{1+2+2-1}{2+2+4} = \frac{1}{2}.$$

Therefore

$$\frac{1}{\left(x^2+\sqrt{2}x+1\right)\left(x^2-\sqrt{2}x+1\right)} = \frac{\frac{1}{2\sqrt{2}}x+\frac{1}{2}}{x^2+\sqrt{2}x+1} + \frac{\frac{-1}{2\sqrt{2}}x+\frac{1}{2}}{x^2-\sqrt{2}x+1}. \tag{8}$$

We can now rewrite (3) using (8):

$$\int \frac{\frac{1}{2\sqrt{2}}x + \frac{1}{2}}{x^2 + \sqrt{2}x + 1} + \frac{\frac{-1}{2\sqrt{2}}x + \frac{1}{2}}{x^2 - \sqrt{2}x + 1}\, dx. \tag{9}$$

Splitting (9) into two integrals yields

$$\int \frac{\frac{1}{2\sqrt{2}}x + \frac{1}{2}}{x^2 + \sqrt{2}x + 1}\, dx + \int \frac{\frac{-1}{2\sqrt{2}}x + \frac{1}{2}}{x^2 - \sqrt{2}x + 1}\, dx. \tag{10}$$

We will factor $\frac{1}{2\sqrt{2}}$ out of both integrals in (10) and then multiply and divide the first integral by 2 and the second integral by -2 which results in

$$\frac{1}{4\sqrt{2}} \int \frac{2x + \sqrt{2} + \sqrt{2}}{x^2 + \sqrt{2}x + 1}\, dx - \frac{1}{4\sqrt{2}} \int \frac{2x - \sqrt{2} - \sqrt{2}}{x^2 - \sqrt{2}x + 1}\, dx. \tag{11}$$

Note: $2\sqrt{2}$ in the first integral was written as $\sqrt{2} + \sqrt{2}$ and $-2\sqrt{2}$ in the second integral was written as $-\sqrt{2} - \sqrt{2}$.

Splitting the fractions in the two integrals in (11) and splitting the two integrals into four integrals yields

$$\frac{1}{4\sqrt{2}} \int \frac{2x + \sqrt{2}}{x^2 + \sqrt{2}x + 1}\, dx + \frac{1}{4\sqrt{2}} \int \frac{\sqrt{2}}{x^2 + \sqrt{2}x + 1}\, dx - \frac{1}{4\sqrt{2}} \int \frac{2x - \sqrt{2}}{x^2 - \sqrt{2}x + 1}\, dx + \frac{1}{4\sqrt{2}} \int \frac{\sqrt{2}}{x^2 - \sqrt{2}x + 1}\, dx. \tag{12}$$

Now we will proceed to solve each of the four integrals individually. In the first integral, note that the numerator is the derivative of the denominator. Thus, the first integral simply becomes

$$\frac{1}{4\sqrt{2}} \int \frac{du}{u} \quad \text{(Where } u = x^2 + \sqrt{2}x + 1 \text{ and } du = 2x + \sqrt{2}\ dx\text{)}.$$

Evaluating that integral and substituting back yields

$$\frac{1}{4\sqrt{2}} \ln\left|x^2 + \sqrt{2}x + 1\right| + C_1.$$

The third integral is really similar. Note that the numerator is the derivative of the denominator. Thus, the third integral becomes

$$\frac{-1}{4\sqrt{2}} \int \frac{du}{u} \quad \text{(Where } u = x^2 - \sqrt{2}x + 1 \text{ and } du = 2x - \sqrt{2}\ dx\text{)}.$$

Evaluating that integral and substituting back yields

$$\frac{-1}{4\sqrt{2}} \ln\left|x^2 - \sqrt{2}x + 1\right| + C_3.$$

The second and fourth integral are also similar. We will multiply and divide both of those integrals by 2 which results in

$$\frac{1}{2\sqrt{2}} \int \frac{\sqrt{2}}{2x^2 + 2\sqrt{2}x + 1 + 1}\, dx \text{ and } \frac{1}{2\sqrt{2}} \int \frac{\sqrt{2}}{2x^2 - 2\sqrt{2}x + 1 + 1}\, dx.$$

Note: 2 in the denominator of both integrals was rewritten as $1 + 1$.

Factoring both denominators yields

$$\frac{1}{2\sqrt{2}}\int \frac{\sqrt{2}}{(\sqrt{2}x+1)^2+1}\,dx \text{ and } \frac{1}{2\sqrt{2}}\int \frac{\sqrt{2}}{(\sqrt{2}x-1)^2+1}\,dx.$$

Now we notice that each integral is of the form $\int \frac{du}{u^2+1}$ which is equal to $\tan^{-1}(u)$. So the second integral in (12) becomes

$$\frac{1}{2\sqrt{2}}\int \frac{du}{u^2+1} \quad \text{(Where } u=\sqrt{2}x+1 \text{ and } du=\sqrt{2}\ dx\text{)}.$$

Evaluating that integral and substituting back yields

$$\frac{1}{2\sqrt{2}}\tan^{-1}\left(\sqrt{2}x+1\right)+C_2.$$

Similarly, the fourth integral becomes

$$\frac{1}{2\sqrt{2}}\int \frac{du}{u^2+1} \quad \text{(Where } u=\sqrt{2}x-1 \text{ and } du=\sqrt{2}\ dx\text{)}.$$

Evaluating that integral and substituting back yields

$$\frac{1}{2\sqrt{2}}\tan^{-1}\left(\sqrt{2}x-1\right)+C_4.$$

Substituting these results into (12) results in

$$\frac{1}{4\sqrt{2}}\ln\left|x^2+\sqrt{2}x+1\right|+C_1+\frac{1}{2\sqrt{2}}\tan^{-1}\left(\sqrt{2}x+1\right)+C_2-\frac{1}{4\sqrt{2}}\ln\left|x^2-\sqrt{2}x+1\right|+C_3$$
$$+\frac{1}{2\sqrt{2}}\tan^{-1}\left(\sqrt{2}x-1\right)+C_4. \tag{13}$$

Now we will combine the constants, combine the logarithms, and factor out $\frac{1}{4\sqrt{2}}$ in (13). Thus we have obtained the solution:

$$\int \frac{dx}{x^4+1}=\frac{\ln\left|\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right|+2\tan^{-1}\left(\sqrt{2}x+1\right)+2\tan^{-1}\left(\sqrt{2}x-1\right)}{4\sqrt{2}}+C$$