Calculus III

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Definition: An ellipse as the locus of all coplanar points whose distance to two fixed points, foci, add to the same constant. An ellipse has an equation of the form:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \text{ where}$$

- (h, k) is the center of the ellipse,
- $(a, 0)$ and $(-a, 0)$ are the horizontal vertices, and
- $(0, b)$ and $(0, -b)$ are the vertical vertices.

Let the point (x, y) lie on the ellipse and the points $(c, 0)$ and $(-c, 0)$ be the two foci. Let the distance between (x, y) and $(c, 0)$ be equal to d_1 and the distance between (x, y) and $(-c, 0)$ be equal to d_2. Then from the definition of an ellipse, $d_1 + d_2 = k$ where k is some constant.

Consider the case when $(x, y) = (a, 0)$. Then $d_1 = a - c$ and $d_2 = a + c$. Then $d_1 + d_2 = (a-c) + (a+c) = 2a$. Because for all points (x, y) on the ellipse, $d_1 + d_2$ is constant, we can call that constant $2a$.

Consider the case when $(x, y) = (0, b)$. Then $d_1 = d_2 = a$ and a right triangle is formed with vertices at $(0, 0), (0, b)$, and $(c, 0)$. The two legs are b and c while the hypotenuse is d_1 which is equal to a. From the Pythagorean Theorem we get that $b^2 = a^2 - c^2$.

Suppose we have an ellipse with the center at $(0, 0)$. Then:

$$d_1 + d_2 = 2a$$

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a$$

$$\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$$

$$(x-c)^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2$$

$$x^2 - 2xc + c^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2cx + c^2 + y^2$$

$$-2xc = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + 2xc$$

$$-4xc - 4a^2 = -4a\sqrt{(x+c)^2 + y^2}$$

$$xc + a^2 = a\sqrt{(x+c)^2 + y^2}$$

$$x^2c^2 + 2xca^2 + a^4 = a^2((x+c)^2 + y^2)$$

$$x^2c^2 + 2xca^2 + a^4 = a^2x^2 + 2xca^2 + a^2c^2 + a^2y^2$$

$$x^2c^2 = a^2x^2 + a^2c^2 + a^2y^2 - a^4$$

$$x^2c^2 = a^2(x^2 + c^2 + y^2 - a^2)$$

$$x^2c^2 = a^2(x^2 + y^2 - (a^2 - c^2))$$

$$x^2c^2 = a^2(x^2 + y^2 - b^2)$$

$$x^2c^2 = a^2x^2 + a^2y^2 - a^2b^2$$

$$x^2c^2 - a^2x^2 = a^2y^2 - a^2b^2$$
$$-x^2(a^2 - c^2) = a^2y^2 - a^2b^2$$
$$-b^2x^2 = a^2y^2 - a^2b^2$$
$$x^2 = \frac{-a^2}{b^2}y^2 + a^2$$
$$\frac{x^2}{a^2} = \frac{-y^2}{b^2} + 1$$
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This is a special case where the center is at $(0, 0)$. A more general form is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ where the center is at (h, k).

Definition: A Hyperpola is the locus of coplanar points whose distances from two fixed points, foci, have a constant difference. A hyperbola has an equation of the form:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \text{ where}$$

- (h, k) is the center of the hyperbola,
- $(a, 0)$ and $(-a, 0)$ are the vertices, and
- b is defined as $\sqrt{c^2 - a^2}$.

Let the point (x, y) lie on the hyperbola and the points $(c, 0)$ and $(-c, 0)$ be the two foci. Let the distance between (x, y) and $(c, 0)$ be equal to d_1 and the distance between (x, y) and $(-c, 0)$ be equal to d_2. Then from the definition of a hyperbola, $d_1 - d_2 = k$ where k is some constant.

Consider the case when $(x, y) = (a, 0)$. Then $d_1 = c - a$ and $d_2 = c + a$. Then $d_1 - d_2 = (c-a) - (c+a) = -2a$. Because for all points (x, y) on the hyperbola, $d_1 - d_2$ is constant, we can call that constant $-2a$.

Because we define b as $\sqrt{c^2 - a^2}$, geometrically it is a the distance between the vertex, $(a, 0)$ and a point on the intersection of the asymptote and the line $x = a$.

Suppose we have a hyperbola with the center at $(0, 0)$. Then:

$$d_1 - d_2 = -2a$$
$$\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = -2a$$
$$\sqrt{(x-c)^2 + y^2} = -2a + \sqrt{(x+c)^2 + y^2}$$
$$(x-c)^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2$$
$$x^2 - 2xc + c^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2cx + c^2 + y^2$$
$$-2xc = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + 2xc$$
$$-4xc - 4a^2 = -4a\sqrt{(x+c)^2 + y^2}$$
$$xc + a^2 = a\sqrt{(x+c)^2 + y^2}$$
$$x^2c^2 + 2xca^2 + a^4 = a^2((x+c)^2 + y^2)$$
$$x^2c^2 + 2xca^2 + a^4 = a^2x^2 + 2xca^2 + a^2c^2 + a^2y^2$$
$$x^2c^2 = a^2x^2 + a^2c^2 + a^2y^2 - a^4$$
$$x^2c^2 = a^2(x^2 + c^2 + y^2 - a^2)$$
$$x^2c^2 = a^2(x^2 + y^2 + (c^2 - a^2))$$

$$x^2c^2 = a^2(x^2 + y^2 + b^2)$$
$$x^2c^2 = a^2x^2 + a^2y^2 + a^2b^2$$
$$x^2c^2 - a^2x^2 = a^2y^2 + a^2b^2$$
$$x^2(c^2 - a^2) = a^2y^2 + a^2b^2$$
$$b^2x^2 = a^2y^2 + a^2b^2$$
$$x^2 = \frac{a^2}{b^2}y^2 + a^2$$
$$\frac{x^2}{a^2} = \frac{y^2}{b^2} + 1$$
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

This is a special case where the center is at $(0, 0)$. A more general form is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ where the center is at (h, k).

There are two asymptotes, lines which the hyperbola approaches but never reaches. In order to find their equations, we solve the equation for y.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$$
$$y^2 = b^2\left(\frac{x^2}{a^2} - 1\right)$$
$$y = \pm b\sqrt{\frac{x^2}{a^2} - 1}$$
$$y = \pm b\sqrt{\frac{a^2}{a^2}\left(\frac{x^2}{a^2} - 1\right)}$$
$$y = \pm b\sqrt{\frac{1}{a^2}\left(x^2 - a^2\right)}$$
$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$$
$$\text{Since, } \lim_{x\to\infty}\sqrt{x^2 - a^2} = \sqrt{x^2} = x,$$
$$\text{The two asymptotes of a hyperbola are } y = \pm\frac{b}{a}x$$