**2011 LSMSA Math Competition**
Calculus Individual

1. Find the equations of lines in $y = mx + b$ form if the lines pass through the point (2,4) and are tangent to the parabola $2x^2 - 3x + 10 = y$.

   Solution: For the two lines to intersect $mx + b = 2x^2 - 3x + 10$. Since the two slopes are equal $m = 4x - 3$. To pass through (2, 4), $4 = 2m + b$. Substituting the third equation into the first, $mx + 4 - 2m = 2x^2 - 3x + 10$. Substituting the second equation in results in $4x^2 - 3x + 4 - 8x + 6 = 2x^2 - 3x + 10$. Simplifying: $x = 0$ and $x = 4$. For $x = 0$, $m = -3$ so $y = -3x + 10$. For $x = 4$, $m = 13$ so $y = 13x - 22$.

2. Evaluate $\lim\limits_{n\to\infty} \frac{5n^6}{(5n+2)^7 - 5n^7}$.

   Solution: Since the exponent of the denominator is greater than the exponent of the numerator, the answer is 0.

3. Evaluate $\lim\limits_{b\to 1^-} \int_0^b \frac{dx}{\sqrt{1-x^2}}$.

   Solution: $\lim\limits_{b\to 1^-} \int_0^b \frac{dx}{\sqrt{1-x^2}} = \lim\limits_{b\to 1^-} [\arcsin(x)]_0^b = \lim\limits_{b\to 1^-} (\arcsin(b) - \arcsin(0)) = \frac{\pi}{2}$.

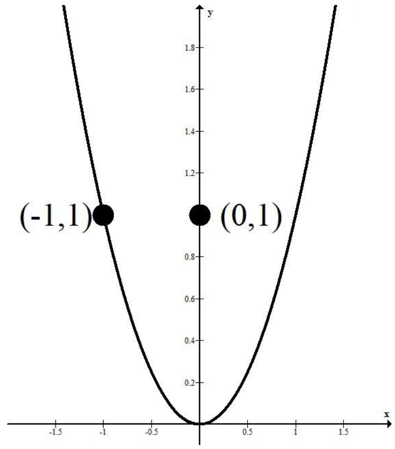
4. Evaluate $\lim\limits_{x\to\infty} x \sin\left(\frac{1}{x}\right)$.

   Solution: $\lim\limits_{x\to\infty} x \sin\left(\frac{1}{x}\right) = \lim\limits_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = 1$.

5. Evaluate $\frac{\frac{d^8}{dx^8}\left[x^8+x^7+x^6\right]}{\frac{d^6}{dx^6}\left[x^6+x^5+x^4\right]}$.

   Solution: $\frac{\frac{d^8}{dx^8}\left[x^8+x^7+x^6\right]}{\frac{d^6}{dx^6}\left[x^6+x^5+x^4\right]} = \frac{8!}{6!} = 56$.

6. Nick Mead rides his bike along a parabolic valley, $y = x^2$, while looking at a babe at point (0,1). He moves with a constant $x$ velocity, $\frac{dx}{dt} = 3$. He starts from the far left of the valley. At what rate, $\frac{d\theta}{dt}$, will he have to move his eyes to keep track of his Goddess's beauty when he is at (-1,1)? That is, assuming he doesn't fall.




Solution: $\tan(\theta) = \frac{y-1}{x} = \frac{x^2-1}{x}$. $\sec^2(\theta)\frac{d\theta}{dt} = \frac{(2x)x+(1-x^2)}{x^2}\frac{dx}{dt} = \frac{x^2+1}{x^2}\frac{dx}{dt}$. $\sec^2(\theta) = \tan^2(\theta)+1 = \left(\frac{x^2-1}{x}\right)^2 + 1$. At $x=-1$, $\sec^2(\theta) = 1$. $\frac{d\theta}{dt} = 3\left(\frac{2}{1}\right) = 6$.

7. Find $f'(2)$ if $f(x) = e^{g(x)}$ and $g(x) = \int_2^x \frac{t}{1+t^4}\,dt$.

   Solution: $f'(x) = g'(x)e^{g(x)}$. $g'(x) = \frac{x}{1+x^4}$. $g'(2) = \frac{2}{17}$ and $g(x) = 0$. $f'(2) = g'(2)e^{g(2)} = \frac{2}{17}e^0 = \frac{2}{17}$.

8. Find $\int_{-50}^{50} \sin(\sin(x))\,dx$.

   Solution: Since $\sin(\sin(x))$ is an odd function, under symmetric limits the integral is 0.

9. In terms of its height $h$ and width $w$, find a formula for the area under a parabola.

   Solution: The formula for the generic parabola is $y = ax(w-x)$. The vertex is at $y = a\left(\frac{w}{2}\right)^2 = h$. $a = \frac{4h}{w^2}$. The area is $A = \int_0^w xw - x^2\,dx = \frac{4h}{w^2}\left[\frac{wx^2}{2} - \frac{x^3}{3}\right]_0^w = \frac{4h}{w^2}\left(\frac{w^3}{2} - \frac{w^3}{3}\right) = \frac{4h}{w^2}\left(\frac{w^3}{6}\right) = \frac{2}{3}hw$.

10. Euler's gamma function $\Gamma(x)$ ("gamma of $x$"; $\Gamma$ is a Greek capital g) uses an integral to extend the factorial function from the nonnegative integers to other real values. The formula is

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt, \qquad x > 0.$$

   If $n$ is a nonnegative integer, $\Gamma(n+1) = n!$

   a. Show that $\Gamma(1) = 1$.

   b. Then apply integration by parts to the integral for $\Gamma(x+1)$ to show that $\Gamma(x+1) = x\Gamma(x)$.

   Solution:
   a.

$$\Gamma(1) = \int_0^\infty t^0 e^{-t}\,dt = \left[-e^{-t}\right]_0^\infty = 0 - (-1) = 1.$$

   b.

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt = -\left[t^x e^{-t}\right]_0^\infty + x\int_0^\infty t^{x-1}e^{-t}\,dt = x\Gamma(x)$$

11. Find the slope of a line tangent to a circle of radius 2 centered at the origin at the point $(2, \theta)$ (in polar coordinates) in terms of $\theta$.

    Solution: $y = 2\sin(\theta)$ and $x = 2\cos(\theta)$. $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = -\frac{2\cos(\theta)}{2\sin(\theta)} = -\cot(\theta)$.

12. Use Leibniz's Rule to find the value of $x$ that maximizes the value of the integral

$$\int_x^{x+3} t(5-t)\,dt.$$

    Solution: $\frac{d}{dx}\int_x^{x+3} t(5-t)\,dt = 0$. $(x+3)(5-x-3) - x(5-x) = 0$ which simplifies to $x = 1$.

13. Evaluate $\lim_{x\to 0^+} (\cos(\sqrt{x}))^{1/x}$.

    Solution: $y = \lim_{x\to 0^+} (\cos(\sqrt{x}))^{1/x}$. Taking the natural logarithm of both sides yields $\ln(y) = \lim_{x\to 0^+} \frac{1}{x}\ln(\cos(\sqrt{x}))$. Using L'Hôpital's Rule results in $\ln(y) = \lim_{x\to 0^+} \frac{-\sin(\sqrt{x})}{2\sqrt{x}\cos(\sqrt{x})}$. Simplifying results in $\ln(y) = -\frac{1}{2}$ or $y = e^{-1/2} = \frac{1}{\sqrt{e}}$.

14. Evaluate (in simplest form) $\frac{d}{dx}\left[\frac{\sin^3(x)}{\cos(x)} + \frac{1}{2}(\sin(2x))\right]$.

    Solution: $\frac{d}{dx}\left[\frac{\sin^3(x)}{\cos(x)} + \frac{1}{2}(\sin(2x))\right] = \frac{d}{dx}\left[\frac{\sin^3(x)}{\cos(x)} + \sin(x)\cos(x)\right] = \frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\left(\sin^2(x) + \cos^2(x)\right)\right] = \frac{d}{dx}\tan(x) = \sec^2(x)$.

15. a. If $\int_0^1 7f(x)\,dx = 7$, does $\int_0^1 f(x)\,dx = 1$?

    b. If $\int_0^1 f(x)\,dx = 4$ and $f(x) \geq 0$, does $\int_0^1 \sqrt{f(x)}\,dx = \sqrt{4} = 2$?

    Solution:

    a. Yes

    b. No