Double Pendulum

Rustem Bilyalov

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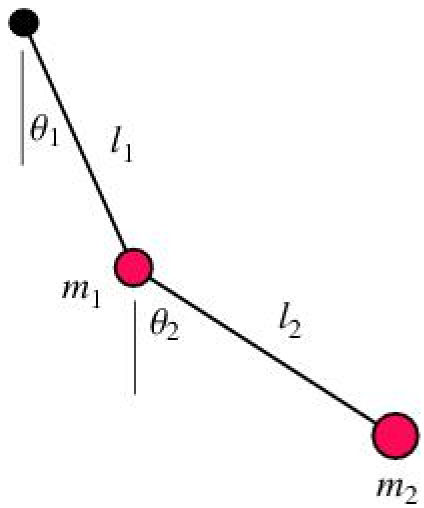

A double pendulum consists of one pendulum attached to another. Consider a double bob pendulum with masses m_1 and m_2 attached by rigid massless wires of lengths l_1 and l_2. Further, let the angles the two wires make with the vertical be denoted θ_1 and θ_2, as illustrated above. Finally, let gravity be given by g. Then the positions of the bobs are given by

$$x_1 = l_1 \sin(\theta_1) \tag{1}$$
$$y_1 = -l_1 \cos(\theta_1) \tag{2}$$
$$x_2 = l_1 \sin(\theta_1) + l_2 \sin(\theta_2) \tag{3}$$
$$y_2 = -l_1 \cos(\theta_1) - l_2 \cos(\theta_2) \tag{4}$$

The potential energy of the system is then given by

$$\begin{aligned} V &= m_1 g y_1 + m_2 g y_2 \\ &= m_1 g(-l_1 \cos(\theta_1)) + m_2 g(-l_1 \cos(\theta_1) - l_2 \cos(\theta_2)) \\ &= -m_1 g l_1 \cos(\theta_1) - m_2 g l_1 \cos(\theta_1) - m_2 g l_2 \cos(\theta_2) \\ &= -(m_1 + m_2) g l_1 \cos(\theta_1) - m_2 g l_2 \cos(\theta_2), \end{aligned} \tag{5}$$

and the kinetic energy by

$$\begin{aligned} T &= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \\ &= \frac{1}{2} m_1 (x_1^2 + y_1^2) + \frac{1}{2} m_2 (x_2^2 + y_2^2) \\ &= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 [l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_1)]. \end{aligned} \tag{6}$$

The Lagrangian is then

$$\begin{aligned} L &\equiv T - V \\ &= \frac{1}{2}(m_1+m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) + (m_1+m_2)g l_1\cos(\theta_1) + m_2 g l_2\cos(\theta_2). \quad (7) \end{aligned}$$

Therefore, for θ_1,

$$\begin{aligned} \frac{\partial L}{\partial \dot{\theta}_1} &= (m_1+m_2)l_1^2\dot{\theta}_1 + m_2 l_1 l_2\dot{\theta}_2\cos(\theta_1-\theta_2) \\ &= m_1 l_1^2\dot{\theta}_1 + m_2 l_1^2\dot{\theta}_1 + m_2 l_1 l_2\dot{\theta}_2\cos(\theta_1-\theta_2) \quad (8) \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) &= (m_1+m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2\ddot{\theta}_2\cos(\theta_1-\theta_2) - m_2 l_1 l_2\dot{\theta}_2\sin(\theta_1-\theta_2)(\dot{\theta}_1-\dot{\theta}_2) \quad (9) \\ \frac{\partial L}{\partial \theta_1} &= -l_1 g(m_1+m_2)\sin(\theta_1) - m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2), \quad (10) \end{aligned}$$

so the Euler-Lagrange differential equation for θ_1 becomes

$$\begin{aligned} (m_1+m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2\ddot{\theta}_2\cos(\theta_1-\theta_2) - m_2 l_1 l_2\dot{\theta}_2\sin(\theta_1-\theta_2)(\dot{\theta}_1-\dot{\theta}_2) & \\ + l_1 g(m_1+m_2)\sin(\theta_1) + m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) &= 0 \\ (m_1+m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2\ddot{\theta}_2\cos(\theta_1-\theta_2) & \\ + m_2 l_1 l_2\dot{\theta}_2^2\sin(\theta_1-\theta_2) + l_1 g(m_1+m_2)\sin(\theta_1) + m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) &= 0. \quad (11) \end{aligned}$$

Diving through by l_1, this simplifies to

$$\begin{aligned} (m_1+m_2)l_1\ddot{\theta}_1 + m_2 l_2\ddot{\theta}_2\cos(\theta_1-\theta_2) & \\ + m_2 l_2\dot{\theta}_2^2\sin(\theta_1-\theta_2) + g(m_1+m_2)\sin(\theta_1) + m_2 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) &= 0. \quad (12) \end{aligned}$$

Similarly, for θ_2,

$$\begin{aligned} \frac{\partial L}{\partial \dot{\theta}_2} &= m_2 l_2^2\dot{\theta}_2 + m_2 l_1 l_2\dot{\theta}_1\cos(\theta_1-\theta_2) \quad (13) \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) &= m_2 l_2^2\ddot{\theta}_2 + m_2 l_1 l_2\ddot{\theta}_1\cos(\theta_1-\theta_2) - m_2 l_1 l_2\dot{\theta}_1\sin(\theta_1-\theta_2)(\dot{\theta}_1-\dot{\theta}_2) \quad (14) \\ \frac{\partial L}{\partial \theta_2} &= m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) - l_2 m_2 g\sin(\theta_2), \quad (15) \end{aligned}$$

so the Euler-Lagrange differential equation for θ_2 becomes

$$\begin{aligned} m_2 l_2^2\ddot{\theta}_2 + m_2 l_1 l_2\ddot{\theta}_1\cos(\theta_1-\theta_2) - m_2 l_1 l_2\dot{\theta}_1\sin(\theta_1-\theta_2)(\dot{\theta}_1-\dot{\theta}_2) & \\ - m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) + l_2 m_2 g\sin(\theta_2) &= 0 \\ m_2 l_2^2\ddot{\theta}_2 + m_2 l_1 l_2\ddot{\theta}_1\cos(\theta_1-\theta_2) - m_2 l_1 l_2\dot{\theta}_1^2\sin(\theta_1-\theta_2) + l_2 m_2 g\sin(\theta_2) &= 0. \quad (16) \end{aligned}$$

Diving through by l_1, this simplifies to

$$m_2 l_2\ddot{\theta}_2 + m_2 l_1\ddot{\theta}_1\cos(\theta_1-\theta_2) - m_2 l_1\dot{\theta}_1^2\sin(\theta_1-\theta_2) + m_2 g\sin(\theta_2) = 0. \quad (17)$$