Derivation of the factor c/4

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2. Show that the proportionality constant in (1-4) is $4/c$. That is, show that the relation between spectral radiancy $R_T(\nu)$ and energy density $\rho_T(\nu)$ is $R_T(\nu)d\nu = (c/4)\rho_T(\nu)d\nu$.
The energy density $\rho_T(\nu)$ is expressed as

$$\rho_T(\nu)d\nu = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{h\nu/kT}-1}d\nu.$$

Let $x = \frac{h\nu}{kT}$ and $dx = \frac{h}{kT}d\nu$. Substitution yields

$$\rho_T(\nu)d\nu = \frac{8\pi k^4T^4}{h^3c^3}\frac{x^3}{e^x-1}dx.$$

Since $\int_0^\infty \frac{x^3}{e^x-1}dx = \frac{\pi^4}{15}$,

$$\rho_T(\nu)d\nu = \frac{8\pi k^4T^4}{h^3c^3}\frac{\pi^4}{15}.$$

Stefan's law states that

$$R_T = \sigma T^4, \sigma \equiv \frac{2\pi^5k^4}{15c^2h^3}.$$

It follows that

$$\rho_T(\nu)d\nu = \frac{8\pi k^4T^4}{h^3c^3}\frac{\pi^4}{15} = \frac{4}{c}\frac{2\pi^5k^4}{15c^2h^3}T^4 = \frac{4}{c}\sigma T^4 = \frac{4}{c}R_T(\nu)d\nu$$

or

$$R_T(\nu)d\nu = \frac{c}{4}\rho_T(\nu)d\nu.$$