Thermodynamics Project

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1 Introduction

The program developed uses Green's Theorem along with the Gaussian Quadrature technique to obtain the area of a shape bound by 4 curves. The user is asked to enter the four functions and their respective inverses into the code of the program (in the .cpp file). He then enters the 4 points of intersection into the program. 8 integrals are then evaluated using Gaussian Quadrature and the area of the shape is displayed.

2 Examples

Below are examples of 4 engines and their solutions both analytically and numerically.

2.1 Engine 1: Carnot

This is the famous Carnot engine. For a monatomic gas the four curves are:

$$
\begin{aligned}
f_1(x) &= \frac{100}{x} \\
f_2(x) &= \frac{10^{\frac{8}{3}}}{x^{\frac{5}{3}}} \\
f_3(x) &= \frac{5 \cdot 20^{\frac{5}{3}}}{30^{\frac{2}{3}} x} \\
f_4(x) &= \frac{5 \cdot 20^{\frac{5}{3}}}{x^{\frac{5}{3}}}
\end{aligned}
$$

and the four intersection points are (Going ccw from the top-right corner) $(20, 5), (10, 10), \left(15, \frac{20^{\frac{5}{3}}}{3 \cdot 30^{\frac{2}{3}}}\right), \left(30, \frac{5 \cdot 20^{\frac{5}{3}}}{30^{\frac{5}{3}}}\right)$. This engine is demonstrated in Figure 1.

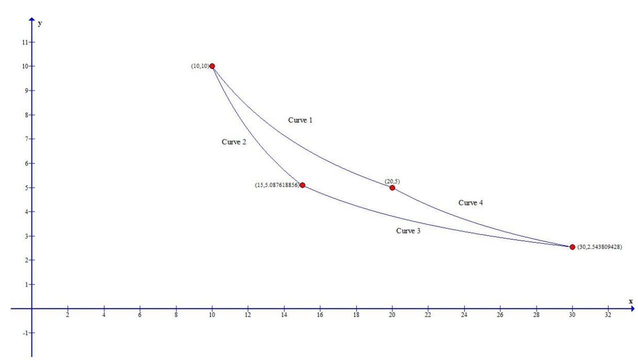


Figure 1: The Carnot Engine

2.1.1 Analytical solution

The areas under the four curves are:

Curve 1:	69.314718
Curve 2:	35.528576
Curve 3:	52.89703
Curve 4:	35.528576

Looking at the figure it is evident that the area under curves 1 and 4 needs to be added and the area under curves 2 and 3 needs to be subtracted:

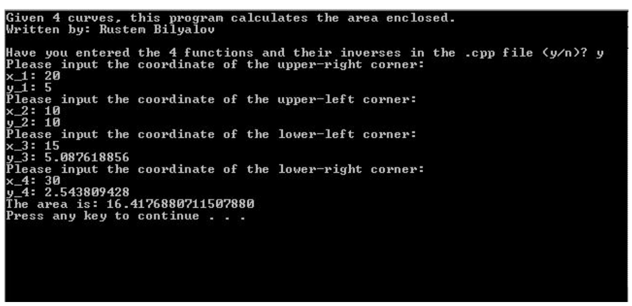
$$A = 69.314718 - 35.528576 - 52.89703 + 35.528576 \approx 16.417688.$$

2.1.2 Numerical solution

The program requires the inverses of the 4 functions:

$$\begin{aligned} g_1(y) &= \frac{100}{y} \\ g_2(y) &= \left(\frac{10^{\frac{8}{3}}}{y}\right)^{\frac{3}{5}} \\ g_3(y) &= \frac{5 \cdot 20^{\frac{5}{3}}}{30^{\frac{2}{3}} y} \\ g_4(y) &= \left(\frac{5 \cdot 20^{\frac{5}{3}}}{y}\right)^{\frac{3}{5}} \end{aligned}$$

Entering these 8 curves and the four intersections into the program yields the following result: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Given 4 curves, this program calculates the area enclosed.
Written by: Rustem Bilyalov

Have you entered the 4 functions and their inverses in the .cpp file (y/n)? y
Please input the coordinate of the upper-right corner:
x_1: 20
y_1: 5
Please input the coordinate of the upper-left corner:
x_2: 10
y_2: 10
Please input the coordinate of the lower-left corner:
x_3: 15
y_3: 5.087618856
Please input the coordinate of the lower-right corner:
x_4: 30
y_4: 2.543809428
The area is: 16.4176880711507880
Press any key to continue . . .
```

Figure 2: Program result for the Carnot Engine

2.2 Engine 2: Diesel

This is the famous Diesel engine. For a monatomic gas the four curves are:

$$\begin{aligned} f_1(x) &= \frac{10 \cdot 20^{\frac{5}{3}}}{x^{\frac{5}{3}}} \\ f_2(x) &= 10 \\ f_3(x) &= \frac{10^{\frac{8}{3}}}{x^{\frac{5}{3}}} \\ f_4 : x &= 30 \end{aligned}$$

and the four intersection points are (Going ccw from the top-right corner) $\left(30, \frac{10 \cdot 20^{\frac{5}{3}}}{30^{\frac{5}{3}}}\right)$, $(20, 10)$, $(10, 10)$, $\left(30, \frac{10^{\frac{8}{3}}}{30^{\frac{5}{3}}}\right)$. This engine is demonstrated in Figure 3.

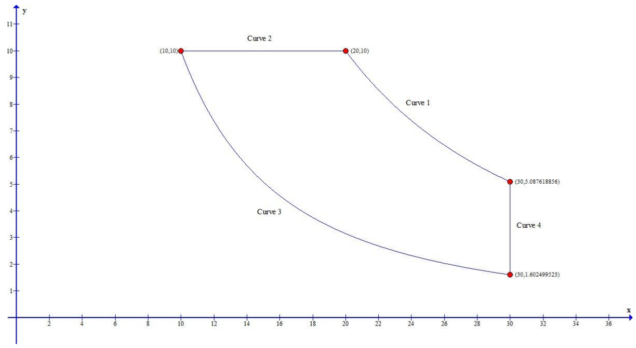

Figure 3: The Diesel Engine

2.2.1 Analytical solution

The areas under the four curves are:

Curve 1:	71.057151
Curve 2:	100
Curve 3:	77.887521
Curve 4:	0

Looking at the figure it is evident that the area under curves 1 and 4 needs to be added and the area under curve 3 needs to be subtracted (the area under curve 4 is 0):

$$A = 71.057151 + 100 - 77.887521 \approx 93.16963.$$

2.2.2 Numerical solution

The program requires the inverses of the 4 functions:

$$\begin{aligned} g_1(y) &= \left(\frac{10 \cdot 20^{\frac{5}{3}}}{y}\right)^{\frac{3}{5}} \\ g_2 : y &= 10 \\ g_3(y) &= \left(\frac{10^{\frac{8}{3}}}{y}\right)^{\frac{3}{5}} \\ g_4(y) &= 30 \end{aligned}$$

Entering these 8 curves and the four intersections into the program yields the following result:

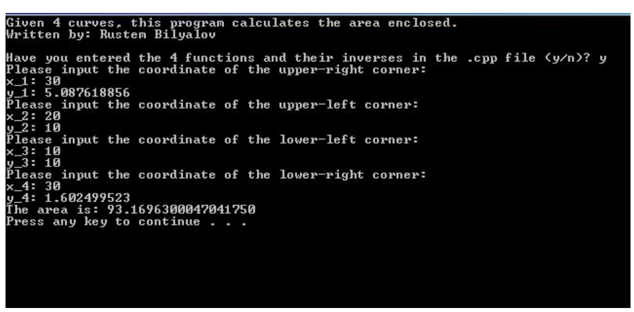

Figure 4: Program result for the Diesel Engine

2.3 Engine 3

This is a very simple engine. The four curves are:

$$\begin{aligned} f_1(x) &= \frac{10}{9}x - \frac{10}{9} \\ f_2(x) &= x \\ f_3(x) &= -\frac{1}{15}x + \frac{16}{3} \\ f_4(x) &= -16x + 324 \end{aligned}$$

and the four intersection points are (Going ccw from the top-right corner) $(19, 20), (10, 10), (5, 5), (20, 4)$. This engine is demonstrated in Figure 5.

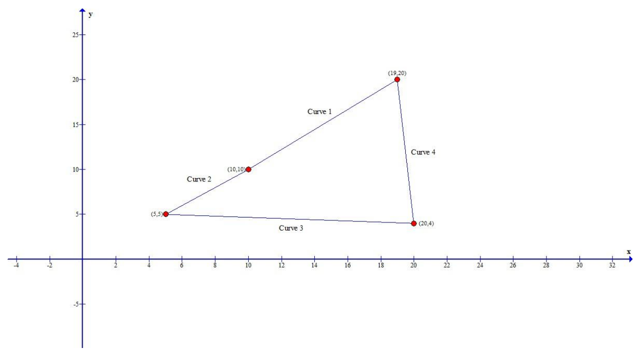

Figure 5: Engine 5

2.3.1 Analytical solution

The areas under the four curves are:

Curve 1:	135
Curve 2:	37.5
Curve 3:	67.5
Curve 4:	12

Looking at the figure it is evident that the area under curves 1, 2, and 4 needs to be added and the area under curve 3 needs to be subtracted:

$$A = 135 + 37.5 - 67.5 + 12 = 117.$$

2.3.2 Numerical solution

The program requires the inverses of the 4 functions:

$$\begin{aligned} g_1(y) &= \frac{9}{10}y + 1 \\ g_2(y) &= y \\ g_3(y) &= -15y + 80 \\ g_4(y) &= \frac{-y}{16} + \frac{81}{4} \end{aligned}$$

Entering these 8 curves and the four intersections into the program yields the following result:

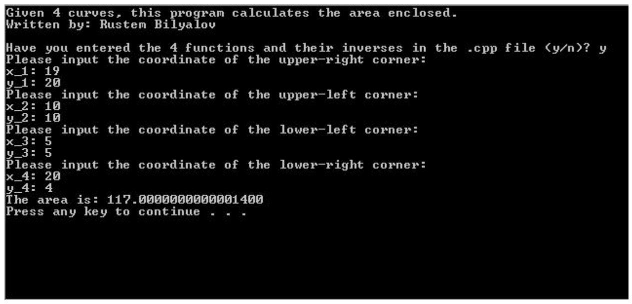

Figure 6: Program result for Engine 3

2.4 Engine 4

This engine is a little more complex. The four curves are:

$$
\begin{aligned}
f_1(x) &= x^2 - \frac{18+\pi^2}{\pi}x + 20 \\
f_2(x) &= \frac{2}{\pi}x \\
f_3(x) &= \sin(x) \\
f_4(x) &= \frac{80\pi^2 - (18+\pi^2)^2}{2\pi(18-\pi^2)}(x-\pi)
\end{aligned}
$$

and the four intersection points are (Going ccw from the top-right corner) $\left(\frac{18+\pi^2}{2\pi}, 20 - \frac{(18+\pi^2)^2}{4\pi^2}\right), (\pi, 2), \left(\frac{\pi}{2}, 1\right), (\pi, 0)$. This engine is demonstrated in Figure 7.

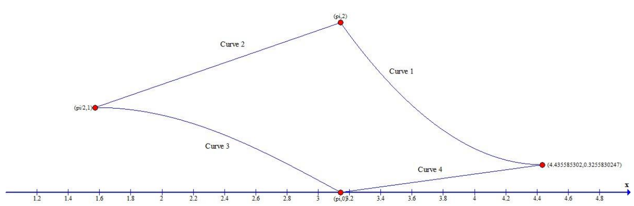

Figure 7: Engine 4

2.4.1 Analytical solution

The areas under the four curves are:

Curve 1:	1.1435	
Curve 2:	2.3562	
Curve 3:	1	
Curve 4:	0.2107	(A TI Calculator was used to obtain these values)

Looking at the figure it is evident that the area under curves 1 and 2 need to be added and the area under curves 3 and 4 needs to be subtracted:

$$A = 1.1435 + 2.3562 - 0.7854 - 0.2107 \approx 2.289.$$

2.4.2 Numerical solution

The program requires the inverses of the 4 functions:

$$
\begin{aligned}
g_1(y) &= \frac{A - \sqrt{A^2 - 4(20 - y)}}{2} \text{ where } A \equiv \frac{18 + \pi^2}{\pi} \\
g_2(y) &= \frac{\pi}{2} y \\
g_3(y) &= \arcsin(y) \\
g_4(y) &= \frac{2\pi(18 - \pi^2)}{80\pi^2 - (18 + \pi^2)^2} y + \pi
\end{aligned}
$$

Care must be taken when entering $g_3(y)$ into the program. We do not want the default range of arcsin so the function must be entered as $g_3(y) = \pi - \arcsin(y)$. Also $g_1(y)$ has a $\pm$ and the correct sign must be chosen. These mistakes can be difficult to catch. Entering these 8 curves and the four intersections into the program yields the following result:

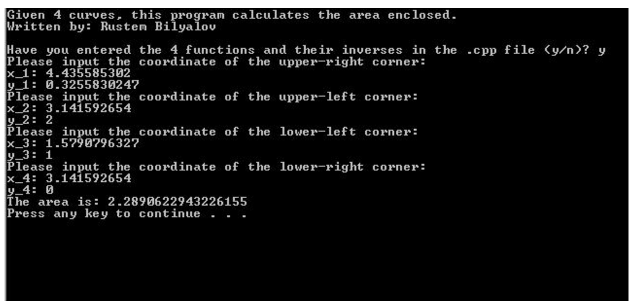

Figure 8: Program result for Engine 4

3 Discussion

3.1 Advantages/Disadvantages

3.1.1 Advantages

The advantage to using Green's Theorem as opposed to straight integration is the fact that with Green's Theorem the program does not have to make "decisions." Obtaining the integrals is not difficult but deciding what to do with them can be. In the four examples above, during the analytical solution, the four integrals were found but a decision was made whether to add or subtract them. With Green's Theorem, such decisions are avoided.

3.1.2 Disadvantages

There are three main problems with this method:

1. 8 integrals have to be approximated as opposed to 4. This doubles the computation time. The fact that this method takes a little longer is not noticeable because Gaussian Quadrature is extremely fast. However, the fact remains that from a pure computational standpoint the method is less efficient than simply approximating the four integrals.

2. Another disadvantage is the fact that the requirement of the inverses of the functions, a restriction is developed: only 1-to-1 functions are accepted by the program. For example in Engine 4, there was a sin function. An engine was picked so the portion of the arcsin was itself a function. Also the quadratic function's inverse can be tricky (careful attention must be paid to which sign must be picked). This can be worked out however it would be difficult.

3. The last major disadvantage not noticeable is the program is applied to thermodynamics but can be a problem in other applications: the shape must exist fully in the first quadrant. However, since the area stays constant no matter where the shape is, it is not a difficult problem to solve.

4 Conclusion

Further improvements on the program would include making the program find the points of intersection itself, inputting a correction to the functions that enables the “functions” to be “relations.” For the sake of experiment, Green’s Theorem was used. If the program was to be expanded into something more generic, it would be replaced with a line integral (which is two times more efficient).

Another thing that can be corrected is allowing the shape to be in other quadrants. This can be achieved by “moving” the shape up and left until it is fully in the first quadrant. This correction is not relevant to Thermodynamics but if the program was to be used for other things, it can be useful.